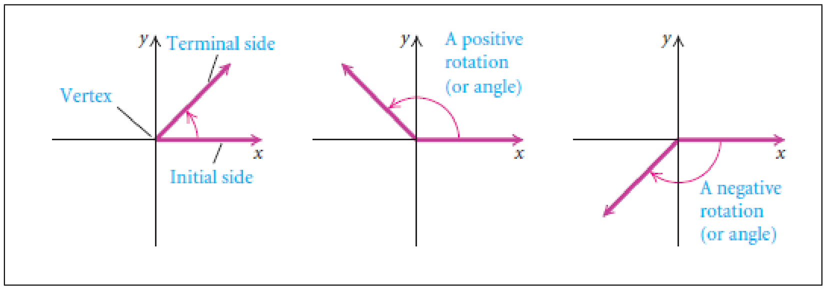
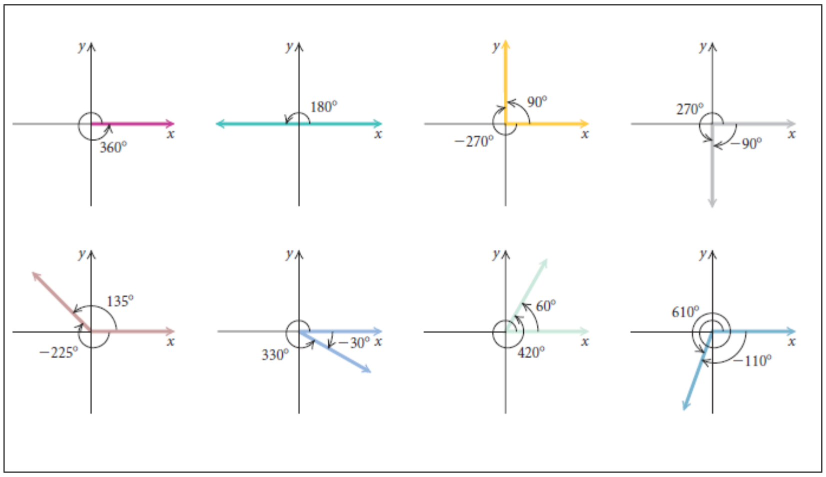
A Quick Review of Trigonometry

As a starting point, we consider a ray with vertex located at the origin whose head is pointing in the direction of the positive real numbers. By rotating the given ray (**initial side**) in a counterclockwise fashion, we obtain a positive angle. The corresponding rotated ray is called the **terminal side** of the angle. In a similar fashion, if we rotate the initial side in a clockwise fashion, we obtain a negative angle as shown below



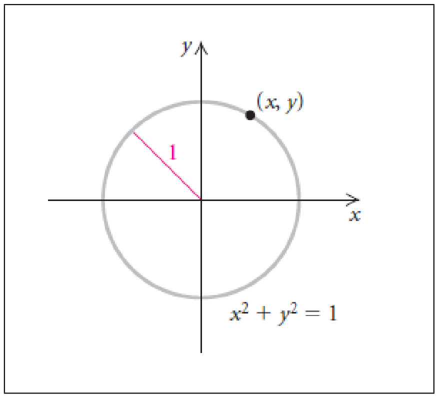
Here are some examples of angles given in degree



We now consider the **unit circle** which is given by the equation

$$x^2 + y^2 = 1.$$

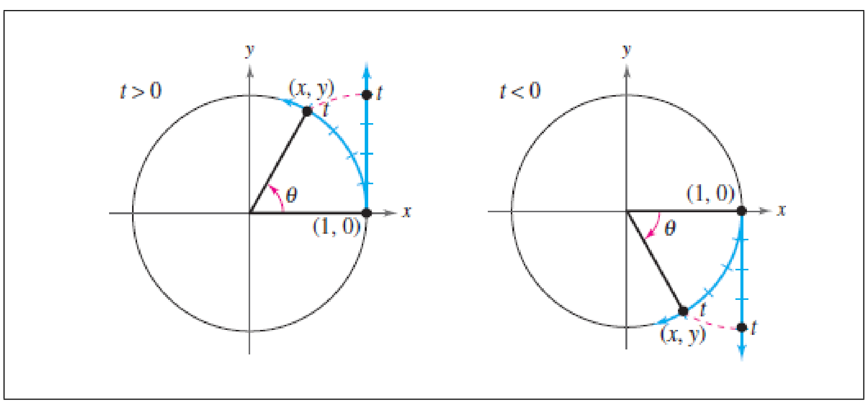
Here is a graph of the unit circle



The circumference of any given circle of radius one is given by $2\pi r$. Therefore, the circumference of the unit circle is equal to

$$2\pi\,(r) = 2\pi\,(1) = 2\pi.$$

Imagine wrapping the real number line around this circle with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping as shown below

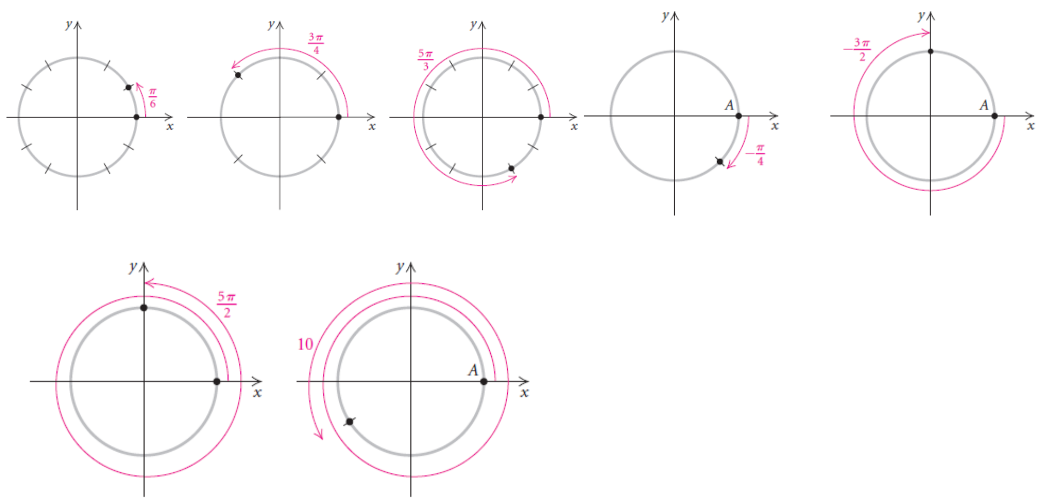


As the real line wraps around the unit circle, each real number t (depicted in the picture above) corresponds to a point (x, y) on the circle.

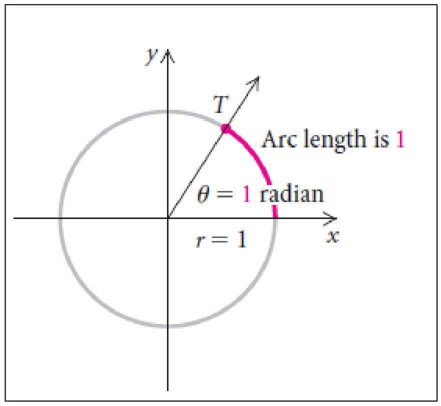
Real Line	**Corresponding points on the circle**
0	$(1, 0)$
2π	$(1, 0)$

Exercise 1 *1. Can you explain why* 2π *corresponds to* $(1,0)$.

2. Describe in your own words, the concepts illustrated in the pictures below



Degree measure is a common unit of angle measure. However, in many scientific fields, the commonly used unit of measure is called the **radian**. Considering the unit circle, let us suppose that we wrap the real line in a counterclockwise fashion by an arc of length 1.



The corresponding angle is by convention

$$\theta = 1 \text{ radian}$$

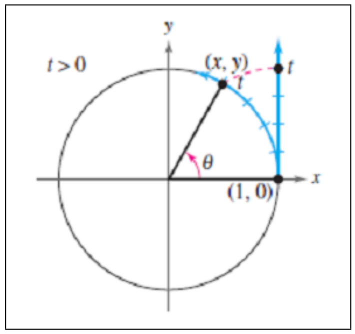
Clearly, it follows that

$$360 \text{ degree } = 2\pi \text{ radian.}$$

Example 2 *Complete the following table*

Degree	*Radian*
150	
	$\frac{\pi}{3}$
	$\frac{2\pi}{5}$

From the discussion above, it clearly follows that the coordinates of the point (x, y) on the unit circle depend on the variable t.



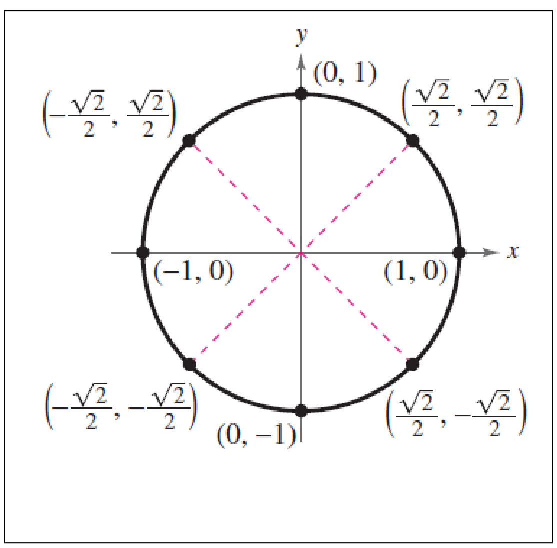
and we define sin, cos, tan, csc, sec and cot as follows:

$$\sin(t) = y \qquad \cos(t) = x \qquad \tan(t) = \tfrac{y}{x}$$
$$\csc(t) = \tfrac{1}{y} \text{ if } y \neq 0 \quad \sec(t) = \tfrac{1}{x} \text{ if } x \neq 0 \quad \cot(t) = \tfrac{x}{y} \text{ if } y \neq 0$$

Let us now divide the unit circle into eight equal arcs. The points on the unit circle corresponding to the t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \text{ and } 2\pi.$$

are shown in the picture below



From the picture shown above, we obtain the following table:

$$\begin{array}{lllll} \cos 0 = 1 & \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} & \cos\frac{\pi}{2} = 0 & \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} & \cos\pi = -1 \\ \sin 0 = 0 & \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} & \sin\frac{\pi}{2} = 1 & \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2} & \sin\pi = -1 \end{array}$$

and

$$\begin{array}{lll} \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} & \cos\frac{3\pi}{2} = 0 & \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2} \\ \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} & \sin\frac{3\pi}{2} = -1 & \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \end{array}$$

In general it is important to be able to know the following

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

Exercise 3 *Without looking at the table above, complete the following table*

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cos								
sin								

Exercise 4 *Without using a calculator, complete the following table by finding the exact values (no approximation)*

	$\frac{7\pi}{6}$	$\frac{15\pi}{4}$	$\frac{7\pi}{2}$
cos			
sin			

Trig Identities

The following are **important** trig identities that we will need to **know !!!**

$$\sin^2(x) + \cos^2(x) = 1 \text{ (Pythagorean Identity)}$$
$$\cos(-x) = \cos x \text{ (cosine is even)}$$
$$\sin(-x) = -\sin x \text{ (sine is odd)}$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) \text{ (double angle formula)}$$
$$\cos(2x) = 1 - 2\sin^2(x) \text{ (double angle formula)}$$
$$\cos 2x = 2\cos^2(x) - 1 \text{ (double angle formula)}$$
$$\sin(2x) = 2\sin x \cos x \text{ (double angle formula)}$$

Example 5 *Use the fact that*

$$\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

to compute

$$\cos\left(\frac{\pi}{12}\right).$$

Solution 6

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(-\frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\sqrt{2}\right) + \left(\frac{1}{2}\sqrt{3}\right)\left(\frac{1}{2}\sqrt{2}\right) \\
&= \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{2}\sqrt{3} \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}.
\end{aligned}$$

Therefore,

$$\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

Inverse Trigonometric Functions

We recall that the trigonometric functions $\sin, \cos, \tan$ are not one-to-one functions. Therefore, they are not invertible. In order to define the inverse of these trigonometric functions, it is important to restrict these functions on some suitable intervals. Here is a table which contains important information regarding trig inverse functions.

Function	**Domain**	**Range**
$y = \sin^{-1}(x) = \arcsin(x)$ where $x = \sin y$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}(x) = \arccos(x)$ where $x = \cos y$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}(x) = \arctan(x)$ where $x = \tan y$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- Let $y = \sin^{-1}(x)$. Then y is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is equal to x
- Let $y = \cos^{-1}(x)$. Then y is the angle located in $[0, \pi]$ whose cosine is equal to x.
- Let $y = \tan^{-1}(x)$. Then y is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is equal to x.

Exercise 7 *Find each of the following exactly in radians*

1.

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right).$$

2.

$$\cos^{-1}\left(-\frac{1}{2}\right).$$

Solution 8 *1. Set*

$$y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right).$$

Then

$$\sin y = \frac{\sqrt{2}}{2} \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

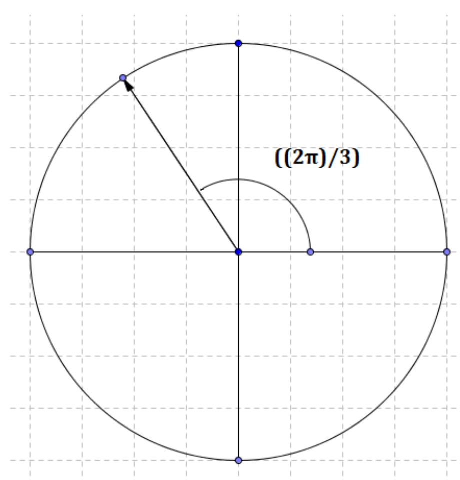
Thus,

$$y = \frac{\pi}{4}.$$

2. Set

$$y = \cos^{-1}\left(-\frac{1}{2}\right).$$

We want to find an angle y whose cosine is equal to $-\frac{1}{2}$ and $y \in [0, \pi]$. Using our unit circle approach we obtain:



and it follows that

$$y = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

Some additional information

FUNCTION	DOMAIN	RANGE
sin	All reals, $(-\infty, \infty)$	$[-1, 1]$
cos	All reals, $(-\infty, \infty)$	$[-1, 1]$
tan	All reals except $k\pi/2$, k odd	All reals, $(-\infty, \infty)$
csc	All reals except $k\pi$	$(-\infty, -1] \cup [1, \infty)$
sec	All reals except $k\pi/2$, k odd	$(-\infty, -1] \cup [1, \infty)$
cot	All reals except $k\pi$	All reals, $(-\infty, \infty)$

INVERSE FUNCTION	DOMAIN	RANGE
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	All reals, or $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\csc^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1}$	All reals, or $(-\infty, \infty)$	$(0, \pi)$