

Understanding Geometric Sums

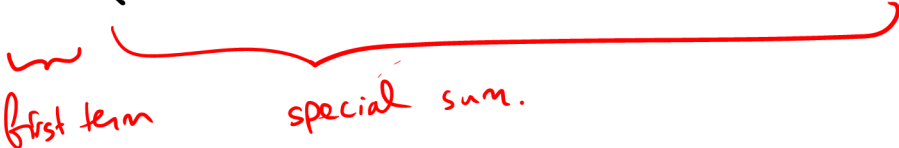
$$a + ar + ar^2 + \dots + ar^{(n-1)}$$


- first term a
- common ratio r
- n terms

$$a + ar + ar^2 + \dots$$

- first term a
- common ratio r

Given any finite geometric sum: (n terms)

$$a + ar + ar^2 + \dots + ar^{(n-1)}$$
$$= a \left(1 + r + r^2 + \dots + r^{(n-1)} \right)$$


Beautiful Fact

for each natural $\#n$

$$\underbrace{1 + r + \dots + r^{(n-1)}}_{\text{LHS}} = \underbrace{\frac{(r^n) - 1}{r - 1}}_{\text{RHS}}$$

happens
to equal

to save time, write

$$\text{LHS} = \underline{S_n} = 1 + r + \dots + r^{\underline{(n-1)}}$$

illustrate this definition

$$S_1 = 1$$

$$S_2 = 1 + r$$

$$S_3 = \boxed{1 + r} + \boxed{r^2} \quad \begin{matrix} \downarrow \\ \text{red arrow} \end{matrix} \quad \begin{matrix} \text{red } 2 = 3-1 \end{matrix}$$

$$S_3 = S_2 + r^{(3)-1}$$

NOTE:

$$S_1 = 1$$

$$\underline{S_{n+1}} = S_n + r^{\underline{(n+1)-1}}$$

called a recursive
definition of S_n

First we'll prove

$$S_n = \frac{r^n - 1}{r - 1}$$

shorthand for

$$1 + r + \dots + r^{n-1} \\ = \frac{r^n - 1}{r - 1}$$

Later we'll use this fact to compute
concrete finite sums.

Proving the Beautiful Fact by Induction

know:

$$S_1 = 1$$

$$S_{k+1} = S_k + r^{(k+1)-1}$$

↗ shorthand for

$$S_{k+1} = \underbrace{1 + r + \dots + r^{k-1}}_{S_k} + \underbrace{r^{(k+1)-1}}_{\text{new term}}$$

show:

$$S_n = \frac{r^n - 1}{r - 1}$$

for all n

these two cases suggest induction:

Base case: $n=1$

Inductive step: $n=k+1$

Base Case: $n=1$

$$\text{LHS} = S_1 = 1 \rightarrow \text{equal!}$$

$$\text{RHS} = \frac{(r)^1 - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$

Inductive Step: $n=k+1$

$$\begin{array}{ccccccc} \square & \square & \dots & \square & \square & \dots \\ \uparrow & & & \uparrow & \uparrow \\ 1 & & & k & k+1 \end{array}$$

suppose: $S_k = \frac{(r)^k - 1}{r - 1}$

show: $S_{k+1} = \frac{(r)^{k+1} - 1}{r - 1}$

Want to show

$$S_n = \frac{(r)^n - 1}{r - 1}$$

for all $n \in \mathbb{N}$

consider the LHS

$$S_{k+1} = S_k + r^{(k+1)-1}$$

$$= \frac{(r)^k - 1}{r - 1} + r^k \cdot \left(\frac{r-1}{r-1} \right)$$

$$= \frac{\cancel{r^k} - 1}{r - 1} + \frac{r^{k+1} \cancel{r^k}}{r - 1}$$

$$\text{so } S_{k+1} = \frac{r^{k+1} - 1}{r - 1}$$

$$S_{k+1} = \underbrace{1 + r + \dots + r^{k-1}}_{S_k} + \underbrace{r^{(k+1)-1}}_{\text{new term}}$$

that's what we wanted to show in inductive step!

base case ✓
inductive step ✓

}

the formula is true
for all $n \in \mathbb{N}$



Using the beautiful fact

Eg: compute $\sum_{i=1}^6 5 \cdot 2^i$

① make this concrete

$$\sum_{i=1}^6 5 \cdot 2^i = \underline{5 \cdot 2} + 5 \cdot 2^2 + \dots + 5 \cdot 2^6$$

② identify a, r, n

$$= 5 \cdot 2 (\underline{1} + 2 + \dots + 2^{\overset{n-1=5}{\underset{n=6}{\textcircled{5}}}})$$

$$\begin{cases} a = 10 \\ r = 2 \\ n = 6 \end{cases}$$

③ use the fact

$$= 10 \cdot \left(\frac{2^6 - 1}{2 - 1} \right)$$

$$= 10 \left(\frac{64 - 1}{2 - 1} \right)$$

$\sum_{i=1}^6 5 \cdot 2^i$	$= 630$
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