Understanding Geometric Sums

$$a + ar + ar^2 + \cdots + ar^{(n-1)}$$

- first term a
- common ratio r
- n terms

$$a + ar + ar^2 + \cdots$$

- first term a
- common ratio r

Given any finite geometric sum: (n terms)

$$a + ar + ar^2 + \cdots + ar^{(n-1)}$$

$$= \underbrace{a}_{\text{first term}} \underbrace{\left(1 + r + r^2 + \cdots + r^{(n-1)}\right)}_{\text{special sum.}}$$

Beautiful Fact

for each natural # n

$$\underbrace{1 + r + \cdots + r^{(n-1)}}_{\text{LHS}} \underset{\text{happens to equal}}{==} \underbrace{\frac{(r^n) - 1}{r - 1}}_{\text{RHS}}$$

to save time, write

LHS: $s_n = 1 + r + \cdots + r^{(n-1)}$

illustrate this definition

$s_1 = 1$

$s_2 = 1 + r$

$s_3 = \boxed{1 + r} + \boxed{r^{2}}$ $\quad 2 = 3-1$

$s_3 = s_2 + r^{(3)-1}$

NOTE:

$s_1 = 1$

$s_{n+1} = s_n + r^{(n+1)-1}$

called a recursive definition of s_n

First we'll prove $s_n == \dfrac{r^n - 1}{r - 1}$

shorthand for

$1 + r + \cdots + r^{n-1} == \dfrac{r^n - 1}{r - 1}$

Later we'll use this fact to compute concrete finite sums.

Proving the Beautiful Fact by Induction

know: $s_1 = 1$

$s_{k+1} = s_k + r^{(k+1)-1}$

shorthand for

$s_{k+1} = \underbrace{1 + r + \cdots + r^{k-1}}_{s_k} + \underbrace{r^{(k+1)-1}}_{\text{new term}}$

show: $s_n == \dfrac{r^n - 1}{r - 1}$ for all n

these two cases suggest induction:

- Base case: $n = 1$
- Inductive step: $n = k+1$

Base Case: $n = 1$

LHS $= s_1 = 1$

RHS $= \dfrac{(r)^1 - 1}{r - 1} = \dfrac{r-1}{r-1} = 1$

equal!

Inductive Step: $n = k+1$

$\square \; \square\square \cdots \square \; \square \cdots$

$\uparrow_1 \quad \uparrow_k \quad \uparrow_{k+1}$

Want to show $S_n = \dfrac{(r)^n - 1}{r - 1}$ for all $n \in \mathbb{N}$

suppose: $s_k == \dfrac{(r)^k - 1}{r - 1}$

show: $s_{k+1} == \dfrac{(r)^{k+1} - 1}{r - 1}$

consider the LHS

$$s_{k+1} = s_k + r^{(k+1)-1}$$

$$= \frac{(r)^k - 1}{r - 1} + r^k \cdot \left(\frac{r-1}{r-1}\right)$$

$$= \frac{\cancel{r^k} - 1}{r - 1} + \frac{r^{k+1} \cancel{- r^k}}{r - 1}$$

so $$s_{k+1} = \frac{r^{k+1} - 1}{r - 1}$$

$s_{k+1} = \underbrace{1 + r + \cdots + r^{k-1}}_{s_k} + \underbrace{r^{(k+1)-1}}_{+\text{ new term}}$

thats what we wanted to show in inductive step!

$$\left.\begin{array}{l}\text{base case } \checkmark \\ \text{inductive step } \checkmark\end{array}\right\} \quad \text{the formula is true for all } n \in \mathbb{N}$$

Using the beautiful fact

Eg: compute $\displaystyle\sum_{i=1}^{6} 5\cdot 2^i$

① make this concrete

$$\sum_{i=1}^{6} 5\cdot 2^i = \underline{5\cdot 2} + 5\cdot 2^2 + \cdots + 5\cdot 2^6$$

② identify a, r, n

$n-1=5$, $n=6$

$$= 5\cdot 2\left(\underline{1} + 2 + \cdots + 2^{5}\right)$$

$$\begin{cases} a = 10 \\ r = 2 \\ n = 6 \end{cases}$$

③ use the fact

$$= 10\cdot\left(\frac{2^6 - 1}{2 - 1}\right)$$

$$= 10\left(\frac{64-1}{2-1}\right)$$

$$\boxed{\sum_{i=1}^{6} 5\cdot 2^i = 630}$$