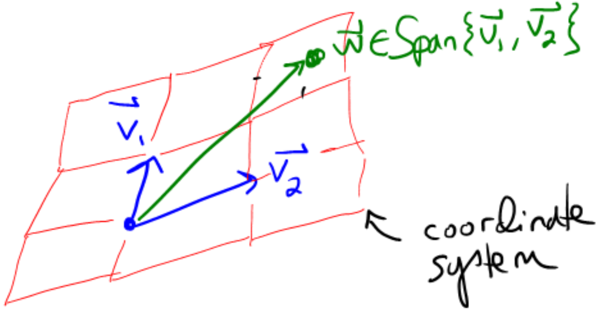
Linear Independence

Recall:

The vectors $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$ generate a coordinate system for $\text{Span}\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$



Eg: $\vec{b} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$

$\vec{b}$'s coordinates WRT $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are 1,2,3

We want an *efficient* coordinate system.

$\vec{b}$ in Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

E.g. Suppose, as above, that $\vec{\mathbf{b}} = \vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2 + 3\vec{\mathbf{v}}_3$.

Suppose also that $\vec{\mathbf{v}}_3 = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2$. } new info

Then $\vec{b} = \vec{v}_1 + 2\vec{v}_2 + 3(\vec{v}_1 + \vec{v}_2)$

so $\vec{b} = 4\vec{v}_1 + 5\vec{v}_2$

so $\vec{b} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$

In this case, $\vec{v}_3$ seems redundant to the coordinate system we can generalize this argument

Lemma:

If $\vec{\mathbf{w}} = r_1\vec{\mathbf{v}}_1 + r_2\vec{\mathbf{v}}_2$

Then $\text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{w}}\} = \text{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$

proof: If $\vec{b} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{w}\}$

then $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{w}$ for $c_1, c_2, c_3 \in \mathbb{R}$

so $= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(r_1\vec{v}_1 + r_2\vec{v}_2)$

$= (\text{some } \#)\vec{v}_1 + (\text{another } \#)\vec{v}_2$

so $\vec{b} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$ ▨ QED.

and the other direction is clear.

want to Define:

"each $\vec{\mathbf{v}}_i$ contributes something new"

$$\iff$$

No $\vec{\mathbf{v}}_i$ is in the span of the other vectors.

Too Many Eqns
need one eqn for each $\vec{v}_i$

Define: $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$ is *linearly independent*

if and only if

the homogeneous equation

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

has ONLY the trivial solution ← the only way to get 0 is $0, 0, \cdots, 0$

Indep $\Leftrightarrow$ $A\vec{x} = \vec{0}$ has only trivial sln.

Fun With Negations

$\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$ is *linearly dependent*

if and only if

they are NOT independent.

if and only if

there is a NON-trivial solution

to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ ← If you can get the $\vec{0}$ with $x_1, \ldots, x_n$ NOT all 0.

E.g. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} -3 \\ 4 \\ 3 \end{bmatrix}$

Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ independent?

E.g. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} 6 \\ -4 \\ -4 \end{bmatrix}$

Is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ independent?

What does a solution *mean*?

Two Observations about Dependence

1. If there is a non-trivial dependence relation, then one vector is in the span of the others

2. If one vector is in the span of the others, then there is a non-trivial dependence relation

Theorem 7: The geometric meaning of dependence

$$\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\} \text{ is Dependent}$$

if and only if

Warning: Not all the $\vec{\mathbf{v}}_i$ will be generated by other $\vec{\mathbf{v}}_j$.

E.g. Consider $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \right\}$