Eg: Solving a system of equations with valid, but not always optimal, row operations

Suppose we have row reduced a system to

$$\left[\begin{array}{ccc|c} 1 & 8 & 3 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

unwisely try setting $r_3^* = r_3 + (-1)\cdot r_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 8 & 3 & 9 \\ 0 & 1 & 0 & 2 \\ -1 & -7 & 0 & -3 \end{array}\right]$$

$$\begin{array}{rcccc} & [\,0 & 1 & 3 & |\ 6\,] \\ + & [\,-1 & -8 & -3 & |\ -9\,] \\ \hline & [\,-1 & -7 & 0 & |\ -3\,] \end{array}$$

set $r_2^* = r_2 + (-1)\, r_3$

$$\left[\begin{array}{ccc|c} 1 & 8 & 3 & 9 \\ 0 & 0 & -3 & -4 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

$$\begin{array}{rcccc} & [\,0 & 1 & 0 & |\ 2\,] \\ + & [\,0 & -1 & -3 & |\ -6\,] \\ \hline & [\,0 & 0 & -3 & |\ -4\,] \end{array}$$

interchange r_2 & r_3

$$\left[\begin{array}{ccc|c} 1 & 8 & 3 & 9 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -3 & -4 \end{array}\right]$$

set $r_2^* = r_2 + r_3$

$$\left[\begin{array}{ccc|c} 1 & 8 & 3 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -4 \end{array}\right]$$

$$\begin{array}{r} [\,0 \quad 1 \quad 3 \mid 6\,] \\ +\ [\,0 \quad 0 \quad -3 \mid -4\,] \\ \hline [\,0 \quad 1 \quad 0 \mid 2\,] \end{array}$$

set $r_1^* = r_1 + (-8)\,r_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -4 \end{array}\right]$$

$$\begin{array}{r} [\,1 \quad 8 \quad 3 \mid 9\,] \\ +\ [\,0 \quad -8 \quad 0 \mid -16\,] \\ \hline [\,1 \quad 0 \quad 3 \mid -7\,] \end{array}$$

set $r_1^* = r_1 + r_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -4 \end{array}\right]$$

$$\begin{array}{r} [\,1 \quad 0 \quad 3 \mid -7\,] \\ +\ [\,0 \quad 0 \quad -3 \mid -4\,] \\ \hline [\,1 \quad 0 \quad 0 \mid -11\,] \end{array}$$

set $r_3^* = \frac{-1}{3} \cdot r_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

original system is equivalent to $\begin{cases} x_1 = -11 \\ x_2 = 2 \\ x_3 = 4/3 \end{cases}$