E.g. Determine h so that the system

$$\begin{cases} x_2 + 5x_3 = 4 \\ x_1 + 4x_2 + 3x_3 = h \\ 2x_1 + 6x_2 - 4x_3 = 2 \end{cases}$$

is consistent

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & 4 \\ 1 & 4 & 3 & h \\ 2 & 6 & -4 & 2 \end{array}\right]$$

$r_1 \leftrightarrow r_2$ $\sim$

$$\left[\begin{array}{ccc|c} \boxed{1} & 4 & 3 & h \\ 0 & 1 & 5 & 4 \\ 2 & 6 & -4 & 2 \end{array}\right] \sim_{r_3 - 2\cdot r_1} \left[\begin{array}{ccc|c} 1 & 4 & 3 & h \\ 0 & 1 & 5 & 4 \\ 0 & -2 & -10 & 2-2h \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & h \\ 0 & 1 & 5 & 4 \\ 0 & -2 & -10 & 2-2h \end{array}\right]$$

$\sim$

$r_3^* = r_3 + 2 \cdot r_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & h \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 2-2h+8 \end{array}\right] \quad 10-2h$$

$r_1^* - 4 \cdot r_2$

$\sim$

$$\left[\begin{array}{ccc|c} 1 & 0 & -17 & h-16 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & \blacksquare\ 10-2h \end{array}\right]$$

← reduced echelon form

By theorem 2:

matrix is consistent $\Leftrightarrow$ reduced echelon form does NOT contain $[0\ 0\ 0 \mid \blacksquare]$ ← nonzero

Reduced echelon form does NOT contain

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & \blacksquare \end{array}\right]$$

$$\Leftrightarrow$$

$$10 - 2h = 0$$

$$\Leftrightarrow$$

$$h = 5$$

has SOME solution

Summary the system is consistent $\Leftrightarrow$ $h = 5$.

NOTICE If the system is consistent $\Rightarrow$ NO pivot in column 3 $\Rightarrow$ $(x_3$ is free) $\Rightarrow$ ∞-many solutions

NOTICE two cases for original system

$h \neq 5 \Rightarrow$ system has NO solutions (inconsistent)

$h = 5 \Rightarrow$ system has ∞-many solutions.