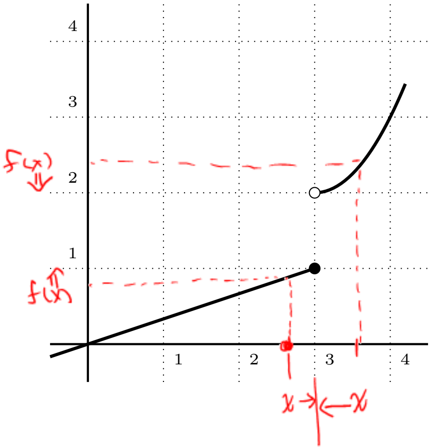
# Review of Limits

Suppose $y = f(x)$ is given by the graph




Compute the limits:

$\lim_{x \to 3^-} f(x) = 1$

$\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x) = \text{DNE}$

undirected

**Idea:**

to "see" $\lim_{x \to a} f(x)$

"grab on" to $x$ near $a$

"pull" $x$ toward $a$

& "look" at where $f(x)$ "goes"

**E.g.** Compute $\lim_{x \to 5} \frac{2x}{3-x}$

think $x$ "just about" 5

$\Rightarrow$ fraction $\approx \frac{2(\text{about } 5)}{3 - \text{about } 5} = \frac{\text{about } 10}{\text{about } -2} = \text{about } -5$

$= -5$

**Two Key Ideas**

- $\dfrac{1}{\text{a smaller positive}}$ is a bigger positive
- $\dfrac{1}{\text{a smaller negative}}$ is a bigger negative

$$\frac{1}{0.01} = 100 \qquad \frac{1}{0.001} = 1000$$

$$\frac{1}{-0.01} = -100 \qquad \frac{1}{-0.001} = -1000$$

**E.g.** $\displaystyle\lim_{x\to 0^+} \frac{1}{x}$

$= \infty$

**E.g.** $\displaystyle\lim_{x\to 0^-} \frac{1}{x}$

$= -\infty$

**E.g.** Compute $\lim_{x \to 3^-} \boxed{\dfrac{2x}{3-x}}$


think $x$ is "just under 3"

$$\text{fraction} \approx \frac{2\,(\text{just under } 3)}{3-(\text{just under } 3)}$$

$$\approx \frac{6}{\text{smaller \& positive}}$$

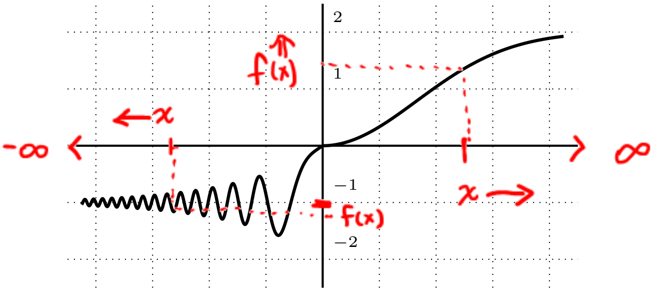
$$\approx \text{BIGGER pos}$$

$3 - 2.99 = 0.01$

$= \infty$

# Limits and Eventual Behavior

Consider the function




Compute the limits:

$$\lim_{x\to\infty} \boxed{f(x)} = 2$$

$$\lim_{x\to-\infty} \boxed{f(x)} = -1$$

to "see" $\lim\limits_{x\to\infty} f(x)$

"grab on" to $x$, move it toward $\infty$

and see where $f(x)$ is going.

## Thinking about Infinite Limits

← think about $x$ being VERY BIG.

$$\lim_{x\to\infty} \frac{(5+x)}{(1+x)}$$

$\nearrow \infty$

$\searrow \infty$

$$\boxed{\begin{array}{l} \underline{\text{think}} \quad x \approx \text{us debt} \\ \text{fraction} \approx \dfrac{5 + \text{us debt}}{1 + \text{us debt}} \approx \dfrac{\text{us debt}}{\text{us debt}} \end{array}}$$

$= 1.$

## Underdetermined Limits

Limits of type $\frac{\infty}{\infty}$ and type $\frac{0}{0}$ are *underdetermined.*

think $\boxed{\frac{2}{3} \cdot \text{us debt} = \text{HUGE pos}}$

$$\lim_{x\to\infty} \boxed{\frac{2x^2}{3x}} = \lim_{x\to\infty} \left[\frac{2}{3} x\right] = \infty$$

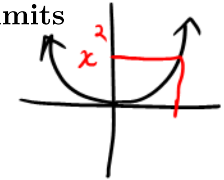
think $\boxed{\frac{2}{3} \cdot \frac{1}{\text{us debt}} \approx 0}$

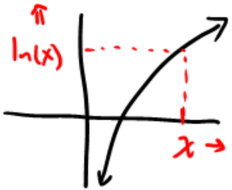
$$\lim_{x\to\infty} \boxed{\frac{2x}{3x^2}} = \lim_{x\to\infty} \left[\frac{2}{3} \cdot \frac{1}{x}\right] = 0$$

$$\lim_{x\to\infty} \boxed{\frac{2x}{3x}} = \lim_{x\to\infty} \left[\frac{2}{3}\right] = \frac{2}{3}$$

**More Complicated Underdetermined Limits**

**E.g.** What is $\lim_{x\to\infty} \boxed{\frac{x \cdot \ln(x)}{x^2}}$ $\begin{matrix}\to \infty \\ \to \infty\end{matrix}$




$$= \lim_{x\to\infty} \boxed{\frac{\ln(x)}{x}} \begin{matrix}\to \infty \\ \to \infty\end{matrix}$$

# L'Hopital's Rule for Underdetermined Fractions

***If*** $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ is type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ ***then***

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{d}{dx}\left[\,f(x)\,\right]}{\frac{d}{dx}\left[\,g(x)\,\right]}$$

as long as the right hand side exists

**E.g.** Compute $\lim_{x\to 1^-} \boxed{\frac{\ln(x)}{x-1}}$ $\to 0$, $\to 0$

$\ln(1) = 0$

$1 - 1 = 0$

$$\overset{H}{=} \lim_{x\to 1^-} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x-1]}$$

$$= \lim_{x\to 1^-} \frac{\frac{1}{x}}{1-0} \quad \frac{1}{x} \to \frac{1}{1}, \quad 1-0 \to 1$$

$$= 1$$

**E.g.** Compute $\lim_{x \to 1^-} \frac{x-1}{2x}$ $\to 0$ $\to 2$

$1-1=0$

$2 \cdot 1 = 2$

NOT type $\frac{0}{0}$ & NOT type $\frac{\infty}{\infty}$

$\Rightarrow$ L'Hopital's does NOT apply

think $x$ just under 1

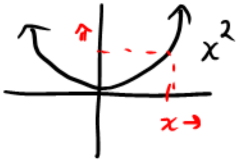
fraction $\approx \frac{\text{small neg}}{2 \cdot (1)}$

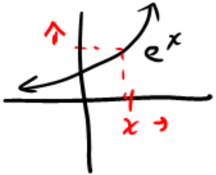
$\approx$ small neg

$x$ 1

$= \frac{0}{2} = 0$

**E.g.** Compute $\lim_{x\to\infty} \boxed{\frac{e^x}{x^2}}$ $\begin{matrix} \to\infty \\ \to\infty \end{matrix}$



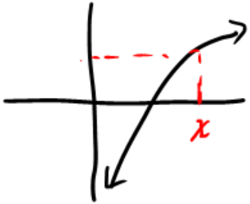



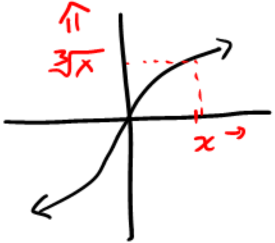

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[x^2]} = \lim_{x\to\infty} \frac{e^x}{2x} \begin{matrix} \to\infty \\ \to\infty \end{matrix}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} \begin{matrix} \to\infty \\ =2 \end{matrix}$$

$$= \infty$$

**E.g.** Compute $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{\sqrt[3]{x}}$ $\begin{matrix}\to\infty \\ \to\infty\end{matrix}$




$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln(x)\right]}{\frac{d}{dx}\left[x^{1/3}\right]}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}\cdot x^{-2/3}} \cdot \frac{x}{x^{1}}$$

Rewrite

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{3}\, x^{1/3}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{3}\sqrt[3]{x}} \quad \to \infty$$

$$= 0$$

# Comparing Rates of Growth

Suppose that $f(x)$ and $g(x)$ both go to $\infty$ as $x \to \infty$.

**Define:** $f$ grows faster than $g$ (written $f \gg g$)

$$\iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \leftarrow \quad \lim_{x\to\infty} \frac{x^2}{x} = \infty$$

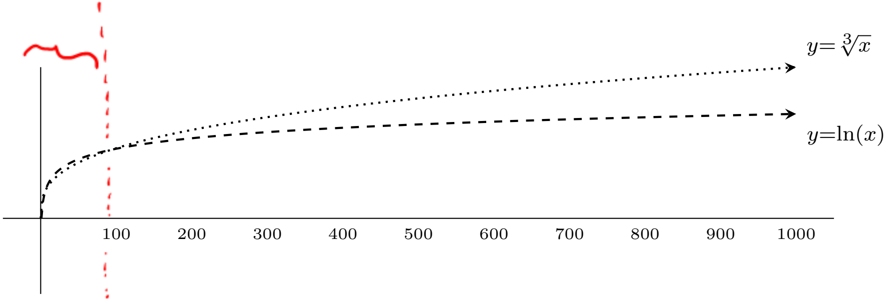
$$\iff \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \leftarrow \quad \lim_{x\to\infty} \frac{x}{x^2} = 0$$



**Eg.** We showed above that $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = 0.$

It follows that $\sqrt[3]{x} \gg \ln(x)$.

To see what $\sqrt[3]{x} \gg \ln(x)$ means, consider the graph of both functions.




Although $\ln(x)$ starts out growing faster,
eventually $\sqrt[3]{x}$ takes over and grows faster in the long run.

Our last example comes from the theory of algorithms.

$n$ = length of your input.

Which grows faster: $n^2$ or $n\log(n)$?

$$\lim_{n\to\infty} \frac{n\cdot\log(n)}{n^2} \quad \begin{matrix}\nearrow\infty\\ \searrow\infty\end{matrix}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{n\cdot\frac{1}{n} + \log(n)\cdot 1}{2n}$$

$$= \lim_{n\to\infty} \frac{\boxed{1+\log(n)}}{\boxed{2n}} \quad \begin{matrix}\nearrow\infty\\ \searrow\infty\end{matrix}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\left.0+\frac{1}{n}\right\}\to 0}{2}$$

$$= 0$$

**Conclude:**

$n^2$ grows faster than $n\cdot\log(n)$