REVERSE MATHEMATICS AND A RAMSEY-TYPE KÖNIG'S LEMMA

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Abstract. In this paper, we propose a weak regularity principle which is similar to both weak König's lemma and Ramsey's theorem. We begin by studying the computational strength of this principle in the context of reverse mathematics. We then analyze different ways of generalizing this principle.

1. Introduction

Weak König's lemma states that for any infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, there is an infinite path p through T. When formalized in second order arithmetic, this theorem is denoted WKL_0. There is a direct correspondence between a path p through a binary tree, which is a function $p : \mathbb{N} \to \{0, 1\}$, and the set $\{x : p(x) = 1\}$. With this in mind, we occasionally identify paths through trees, colorings of singletons, and subsets of $\mathbb{N}$.

Ramsey's theorem is a generalization of the pigeon-hole principle. Given $n \in \mathbb{N}$, let $[\mathbb{N}]^n$ denote the collection of size n subsets of $\mathbb{N}$. The infinite version of Ramsey's theorem says that for every $n, k \in \mathbb{N}$ and every function (called a coloring) $f : [\mathbb{N}]^n \to \{0, \ldots, k-1\}$, there is an infinite set $H \subseteq \mathbb{N}$ which is given one color by f. In the context of second order arithmetic, this principle is denoted RT^n_k (with n, k as above).

Weak König's lemma and Ramsey's theorem can be thought of as asserting the existence of different types of regularity. Viewed topologically, weak König's lemma is essentially the statement "$2^{\mathbb{N}}$ is compact." This carries over to reverse mathematics, where WKL_0 is equivalent to many theorems about compactness (see [13]). For its part, Ramsey's theorem says that no matter how badly behaved a coloring is, it always has a sizable homogeneous set. In the words of T.S. Motzkin, absolute disorder is impossible.

In this paper, we study the computational and reverse mathematical strength of a regularity principle which combines features of weak König's lemma and Ramsey's theorem. We will refer to the statement "for each infinite binary tree T, there is an infinite set H homogeneous for a path through T" as a Ramsey-type König's lemma. In statement 2, we give a formal definition of our Ramsey-type König's

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lemma, denoted RKL, in terms of finite strings. It is the novelty of this principle, rather than the complexity of the proofs, that is the main innovation of this paper.

We begin by showing that RKL is a nontrivial weakening of WKL_0 and of RT^2_2. More formally, we show that SRT^2_2 implies RKL, that WKL_0 implies RKL, and that RKL implies DNR (unless specified, we always work over RCA_0). Applying results of [1] and [11], we conclude that RKL is strictly weaker than WKL_0 and SRT^2_2.

In the remaining sections, we state analogs of RKL for trees generated by infinite sets of strings ($\mathsf{RKL}^{(1)}$) and for arithmetically-definable trees ($\mathsf{RKL}^{(<\omega)}$). We then study the strength of each principle, and obtain the surprising result that these stronger principles are more closely related to RT^2_2 than to WKL_0.

We show that RT^2_2 implies $\mathsf{RKL}^{(1)}$, and that $\mathsf{RKL}^{(1)}$ implies SRT^2_2. By the main results of [5] and [11], it follows that $\mathsf{RKL}^{(1)}$ and WKL_0 are incomparable. We also show that RT^2_2 does not imply $\mathsf{RKL}^{(<\omega)}$ and, by using a result of [11], we show that $\mathsf{RKL}^{(<\omega)}$ does not imply WKL_0. We leave open whether $\mathsf{RKL}^{(<\omega)}$ implies RT^2_2. We summarize our results in figure 1.

1.1. **Working in second order arithmetic.** We assume that the reader is familiar with the basic definitions and results of computability theory and reverse mathematics. For an introduction to reverse mathematics, see chapter I of [13]. For an introduction to computability theory, see part A of [14].

Some care is required to formalize RKL in RCA_0. Our goal here is to study the computational complexity of the homogeneous set H, not of the path p. While there are computable trees T such that each path through T is reasonably complicated, there are paths p with relatively simple homogeneous sets H. We begin with definitions that allow us to say that H is homogeneous for *some* path through T without explicit reference to a path p.

Definition 1 (RCA_0)**.** *H is homogeneous for $\sigma \in 2^{<\mathbb{N}}$ with color $c \in \{0,1\}$* if $\sigma(x) = c$ for each $x \in H$ s.t. $x < |\sigma|$. H is *homogeneous for a path through T* if $\exists c \in \{0,1\}$ s.t. H is homogeneous for σ with color c for arbitrarily long $\sigma \in T$.

Statement 2 (RCA_0)**.** RKL asserts that "for each infinite binary tree T, there is an infinite set H which is homogeneous for a path through T."

Unless stated otherwise, all strings and trees we consider will be binary ($\{0,1\}$-valued). Given $\tau, \sigma \in 2^{<\mathbb{N}}$ we write $\tau \preceq \sigma$ if τ is an initial segment of σ. We write $\sigma \restriction t$ (or $p \restriction t$) to denote the initial segment of $\sigma \in 2^{<\mathbb{N}}$ (or $p \in 2^{\mathbb{N}}$) of length t.

2. Reverse mathematics of RKL

Theorem 3 (RCA_0)**.** *WKL_0 implies RKL.*

Proof. Given an infinite binary tree T, let p be an infinite path through T. Note that $p : \mathbb{N} \to \{0,1\}$ maps singletons into two colors. Applying RT^1_2, which is provable in RCA_0, yields a set H which is homogeneous for p. In particular, $p \restriction t \in T$ and H is homogeneous for $p \restriction t$ for each $t \in \mathbb{N}$. Thus H satisfies definition 1, as desired. □

Definition 4 (RCA_0)**.** A coloring $f : [\mathbb{N}]^2 \to \{0,1\}$ is *stable* if for each x, there is some $t > x$ s.t. $(\forall y > t)[f(x,y) = f(x,t)]$. SRT^2_2 is the theorem "every stable coloring of pairs into two colors has an infinite homogeneous set."

Theorem 5 (RCA_0)**.** *SRT^2_2 implies RKL.*

Proof. Given an infinite tree T, we define a coloring $f : [\mathbb{N}]^2 \to \{0,1\}$ as follows. For each y, let σ_y be the lexicographically least element of T of length y. For each $x < y$, define $f(x,y) = \sigma_y(x)$.

We now show that f is a stable coloring. Fix x, and let T^{ext} denote the strings in T that are extended by arbitrarily long strings in T. For each $\tau \in T - T^{ext}$ of length $x+1$, there is a bound on the length of strings in T extending τ, so there is a least such bound s_τ. Note that s_τ is Δ^0_1 definable (with parameters) from τ. By Σ^0_1 induction, there is a uniform upper bound t for $\{s_\tau : \tau \in 2^{x+1} \wedge \tau \in T - T^{ext}\}$. By Π^0_1 induction, there is a lexicographically least element $\tau_{x+1} \in T^{ext}$ of length $x+1$. Then for each $y > t$, $\sigma_y \restriction (x+1) = \tau_{x+1}$ hence $f(x,y) = \tau_{x+1}(x)$. In general, for each x, $(\exists t)(\forall y > t)[f(x,y) = \tau_{x+1}(x)]$. In other words, f is a stable coloring.

Suppose that H is homogeneous for f with color $c \in \{0,1\}$. We now show that H is homogeneous for a path through T. Fix $t \in \mathbb{N}$. Because H is an infinite set, there is an element $y \in H$ with $y \geq t$. By the definition of f, $(\forall x < y)[\sigma_y(x) = f(x,y)]$. Because H is homogeneous for f with color c and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. Then H is homogeneous for $\sigma_y \in T$ with color c. Since t is arbitrary and $|\sigma_y| \geq t$, we have satisfied definition 1. □

Corollary 6 (RCA_0)**.** RKL *does not imply* SRT^2_2 *or* WKL_0*.*

Proof. By the main result of [11], SRT^2_2 does not imply WKL_0. Because SRT^2_2 implies RKL, RKL cannot imply WKL_0. Similarly, RKL cannot imply SRT^2_2 over RCA_0 because WKL_0 does not imply SRT^2_2 (by Theorem 3.3 of [12] and Theorem 10.5 of [1]). □

We conclude our analysis of RKL by showing that it is not provable in RCA_0, and by showing that RKL is strong enough to imply DNR. When $T \subseteq 2^{<\mathbb{N}}$ is a tree, $[T] \subseteq 2^{\mathbb{N}}$ will be the set of infinite paths through T. The following lemma follows from the proof of lemma 2 in [10].

Lemma 7 (Jockusch, [10])**.** *There is an infinite computable tree T such that for any $p \in [T]$ and for any $e \in \mathbb{N}$, if $|W_e| \geq e+3$ then W_e is not homogeneous for p. In fact, for each $e \in \mathbb{N}$, there is a $t \in \mathbb{N}$ s.t. if $|W_e| \geq e+3$ then W_e is not homogeneous for any string in T of length greater than t.*

A simple corollary is that $\mathsf{RCA}_0 \nvdash \mathsf{RKL}$, via an ω-model. Adapting the proof of Theorem 2.3 from [8], we can obtain a slightly stronger result. We say that a function f is *diagonally non-computable* relative to X if $f(e) \neq \Phi^X_e(e)$ for each e s.t. $\Phi^X_e(e)\downarrow$. The principle DNR asserts that for any set parameter X, there is a function that is diagonally non-computable relative to X.

Theorem 8 (RCA_0)**.** RKL *implies* DNR*.*

Proof. We work relative to a set parameter X. The proof of lemma 2 of [10] (lemma 7 above) works in RCA_0. Let T be the tree defined in this proof. By RKL, there is a set H homogeneous for a path through T. Note that there is a Δ^0_1 definable function $g : \mathbb{N} \to \mathbb{N}$ such that $W_{g(e)}$ is the least $e+3$ elements of H (in the $\mathbb{N}$ order).

We now show that g is a fixed point free function. If $|W_e| < 2^{e+1}$, then $|W_{g(e)}| \neq |W_e|$. Suppose that $|W_e| \geq 2^{e+1}$. By the above lemma, there is some t s.t. W_e is not homogeneous for any $\sigma \in T$ of length greater than t. Because $W_{g(e)} \subset H$ and because H is homogeneous for a path through T, $W_{g(e)}$ is homogeneous for arbitrarily long $\sigma \in T$. In particular $W_e \neq W_{g(e)}$. In other words, g is fixed point free, so can be used to give a Δ^0_1 definition for a DNR function (formalize V.5.8 of [14] in RCA_0). □

Question 9. Does DNR imply RKL?

A number of principles are known to be stronger than DNR, such as ASRAM and ASRT^2_2 from [7]. Proving that one of these principles does not imply RKL would separate DNR from RKL.

3. Trees generated by sets of strings

Definition 10. Given an infinite set of strings Σ, let T_Σ denote the downward closure of Σ. More formally $T_\Sigma = \{\tau : (\exists\sigma \in \Sigma)[\tau \preceq \sigma]\}$.

Statement 11 (RCA_0)**.** $\mathsf{RKL}^{(1)}$ asserts that "for each infinite set of strings Σ, there is a set H which is homogeneous for a path through T_Σ."

Note that if Σ is an infinite computable set of strings, T_Σ is an infinite c.e. tree. In [6], Downey and Jockusch note that each infinite $\Pi^{0,\emptyset'}_1$-class can be generated by a c.e. tree. We extend this slightly, to further motivate our definition of $\mathsf{RKL}^{(1)}$.

Proposition 12. *If T is an infinite Π^0_2 tree, then there is an infinite computable set of strings Σ s.t. $[T] = [T_\Sigma]$. Furthermore, Σ can be taken to contain exactly one string of each length.*

Proof. It suffices to consider Σ^0_1 trees. To see this, suppose that T is Π^0_2. Then there is a formula ϕ which is Δ^0_1 s.t. $\tau \in T \;\leftrightarrow\; (\forall y)(\exists z)\phi(\tau, y, z)$. Using the Δ^0_1 formula $\psi(\tau, \hat{z}) =_{def} (\forall x, y \leq |\tau|)(\exists z < \hat{z})\phi(\tau \upharpoonright x, y, z)$, we can define a Σ^0_1 tree S by $\tau \in S \leftrightarrow (\exists\hat{z})\psi(\tau, \hat{z})$. Then $[S] = \{f : (\forall w)(\exists\hat{z})\psi(f \upharpoonright w, \hat{z})\} = \{f : (\forall x)(\forall y)(\exists z)\phi(f \upharpoonright x, y, z)\} = [T]$, so we may work with S instead.

Given a Σ^0_1 tree T, fix a computable enumeration $\{T_s\}$ of T. If necessary, we computably modify the enumeration to ensure that no τ enters T_s until $s \geq |\tau|$ and that exactly one string enters T at each stage. We computably enumerate the elements of Σ in increasing order. At stage $s > 0$, find $\tau \in T_s - T_{s-1}$, take one $\sigma \succeq \tau$ with $|\sigma| = s$ (the specific choice is not important), and put σ into Σ. It is not difficult to show that $T \subseteq T_\Sigma$ and that $T^{ext}_\Sigma \subseteq T^{ext}$. It follows that $[T] = [T_\Sigma]$. □

We now examine the strength of $\mathsf{RKL}^{(1)}$.

Theorem 13 (RCA_0)**.** RT^2_2 *implies* $\mathsf{RKL}^{(1)}$*.*

Proof. Fix an infinite set of strings Σ. For each y, let $l \geq y$ be the length of the shortest string in Σ of length at least y. Let σ_y be the lexicographically least string in Σ of length l. We now define a coloring $f : [\mathbb{N}]^2 \to \{0, 1\}$ as before. For each $x < y \in \mathbb{N}$, set $f(x, y) = \sigma_y(x)$. Note that f is Δ^0_1-definable.

By RT^2_2, there is an infinite set H homogeneous for f with color $c \in \{0, 1\}$. We claim that H is homogeneous for a path through T_Σ.

Fix $t \in \mathbb{N}$. Because H is infinite, there is some $y \in H$ with $y \geq t$. By definition of f, $(\forall x < y)[f(x, y) = \sigma_y(x)]$. Because H is homogeneous for f with color c and because $y \in H$, $(\forall x < y)[x \in H \implies \sigma_y(x) = c]$. In other words, H is homogeneous for $\sigma_y \upharpoonright y$ with color c. Because $\sigma_y \upharpoonright y \in T_\Sigma$ and because $y \geq t$ with t arbitrary, H is homogeneous for a path through T_Σ (in the sense of definition 1). □

Remark 14. The coloring defined in the proof of theorem 13 is not (necessarily) stable because it is defined in terms of Σ, and not in terms of T. For example, suppose that $\sigma(0) = 0$ for even length strings $\sigma \in \Sigma$, and that $\sigma(0) = 1$ for odd length strings. Then $\lim_y f(0, y)$ does not exist.

There is a natural correspondence between computable colorings $f : [\mathbb{N}] \to \{0, 1\}$ and computable sets $\Sigma \subset 2^{<\mathbb{N}}$ that contain exactly one string of each length. Given f, simply define $\Sigma = \{\sigma_y : \sigma_y \in 2^y \ \wedge \ (\forall x < y)[\sigma_y(x) = f(x, y)]\}$. Using the induced tree T_Σ, it is not difficult to show that $\mathsf{RKL}^{(1)}$ implies SRT^2_2 over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$. Yokoyama was able to eliminate the use of $\mathrm{B}\Sigma^0_2$ by introducing the following principle.

Statement 15 (RCA_0)**.** P^2_2 asserts that "for any Π^0_2-definable subsets A_0, A_1 of $\mathbb{N}$ s.t. $A_0 \cup A_1 = \mathbb{N}$, there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A_0$ or $H \subseteq A_1$."

The principle P^2_2 is particularly useful because it implies the better understood principle D^2_2. This is a special instance of the principle D^n_2, which we will return to in the next section.

Statement 16 (RCA_0)**.** For each $n \in \omega$, D^n_2 asserts that "for each Δ^0_n-definable subset A of $\mathbb{N}$, there exists an infinite set $H \in S(\mathcal{M})$ s.t. $H \subseteq A$ or $H \subseteq \overline{A}$."

Theorem 17 (Cholak, Chong, Jockush, Lempp, Slaman, Yang [1, 4])**.** *Over* RCA_0*,* D^2_2 *is equivalent to* SRT^2_2*.*

Theorem 18 (Yokoyama [15])**.** $\mathsf{RKL}^{(1)}$ *implies* P^2_2*, and hence* SRT^2_2*, over* RCA_0*.*

Proof. Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \mathsf{RCA}_0 + \mathsf{RKL}^{(1)}$ and suppose that A_0, A_1 are Π^0_2-definable subsets of $\mathbb{N}$ s.t. $A_0 \cup A_1 = \mathbb{N}$. We will define a Δ^0_1 function $f : [\mathbb{N}]^2 \to \{0, 1\}$ s.t. if $f(x, y) = i$ for infinitely many y, then $x \in A_i$.

Fix two Σ^0_0 formulas $\theta_i(x, m, n)$ s.t. $x \in A_i \iff (\forall m)(\exists n)\theta_i(x, m, n)$. Using these formulas, we define a helper function:

$$h(x, y) = (\mu z)[(\forall m < y)(\exists n < z)[\theta_0(x, m, n)] \vee (\forall m < y)(\exists n < z)[\theta_1(x, m, n)]].$$

Clearly, h is a Δ^0_1 function.

We must verify in RCA_0 that h is total. Fix $x \in \mathbb{N}$ arbitrary. Then $x \in \mathbb{N} = A_0 \cup A_1$, so $x \in A_0$ or $x \in A_1$. So $(\forall m)(\exists n)\theta_0(x, m, n)$ or $(\forall m)(\exists n)\theta_1(x, m, n)$. Let $y \in \mathbb{N}$ be arbitrary. Suppose $x \in A_i$. Then $(\forall m)(\exists n)\theta_i(x, m, n)$, so clearly $(\forall m < y)(\exists n)\theta_i(x, m, n)$. By $\mathrm{B}\Sigma^0_1$, there is a z_i s.t. $(\forall m < y)(\exists n < z_i)\theta_i(x, m, n)$. Thus, h will find a least z s.t. the desired condition holds.

Define $f(x, y) = 0$ if $(\forall m < y)(\exists n < h(x, y))[\theta_0(x, m, n)]$, and $f(x, y) = 1$ otherwise. Clearly, f is total since h is total, and f is Δ^0_1 since h is total and Δ^0_1. If $f(x, y) = i$ for infinitely many y, then our defn of $h(x, y)$ confirms that $x \in A_i$.

Using f, let $\Sigma = \{\sigma_y : \sigma_y \in 2^y \ \wedge \ (\forall x < y)[\sigma_y(x) = f(x, y)]\}$ and define T_Σ as before. Take H homogeneous for a path through T_Σ with some color $c \in \{0, 1\}$.

Let $x \in H$ be arbitrary. By definition of "homogeneous for a path through T_Σ with color c," there are infinitely many y s.t. $f(x, y) = \sigma_y(x) = c$. By our choice of f, this means that $x \in A_c$. In other words, $H \subseteq A_c$ is the desired infinite set. □

Question 19 (Yokoyama [15])**.** Does D^2_2 imply P^2_2? Does P^2_2 imply $\mathsf{RKL}^{(1)}$?

Corollary 20. $\mathsf{RKL}^{(1)}$ *is incomparable with* WKL_0 *over* RCA_0

Proof. Because WKL_0 does not imply SRT^2_2 (Theorem 3.3 of [12] and Theorem 10.5 of [1]), WKL_0 does not imply $\mathsf{RKL}^{(1)}$. By the main result of [11], RT^2_2 does not imply WKL_0, so $\mathsf{RKL}^{(1)}$ cannot imply WKL_0. □

Remark 21. Using the above arguments, we can rephrase RT^2_2 as the statement "for each Σ which contains exactly one string of each length, there is an infinite H which is homogeneous (with fixed color c) for each $\sigma \in \Sigma$ s.t. $|\sigma| \in H$."

Question 22. Does $\mathsf{RKL}^{(1)}$ imply COH, CAC, ADS or RT^2_2? One implication holds if and only if all implications hold. Does SRT^2_2 imply $\mathsf{RKL}^{(1)}$?

4. Arithmetically-definable trees

Statement 23 (RCA_0)**.** $\mathsf{RKL}^{(<\omega)}$ is the axiom scheme which, for each arithmetic formula ϕ, asserts that "if ϕ defines a tree T containing arbitrarily long strings, there is an infinite set H which is homogeneous for a path through T."

Theorem 24. *Over* RCA_0*, we have the following strict implications:* $\mathsf{ACA}_0 \implies \mathsf{RKL}^{(<\omega)} \implies \mathsf{RKL}^{(1)} \implies \mathsf{RKL}$*.*

The implications are immediate. We have already seen that the third implication is strict. We now show that the first two implications are also strict. We first use the following result from [11] to separate $\mathsf{RKL}^{(<\omega)}$ from ACA_0.

Theorem 25 (Liu, [11])**.** *For every $C \not\geq \emptyset$ and every coloring $p : \mathbb{N} \to \{0, 1\}$, there exists an infinite set H homogeneous for p such that $H \oplus C \not\geq \emptyset$.*

Corollary 26. *There is an ω-model of* $\mathsf{RKL}^{(<\omega)}$ *where* WKL_0 *fails.*

Proof. To build an ω-model $\mathcal{M} = (\omega, S(\mathcal{M}))$ of $\mathsf{RKL}^{(<\omega)}$, we begin with $S(\mathcal{M}) = REC$ and add sets to $S(\mathcal{M})$.

The general strategy for creating a model of $\mathsf{RKL}^{(<\omega)}$ uses a list of the infinite trees which are arithmetically-definable from any set $X \in S(\mathcal{M})$. For each $i \in \mathbb{N}$, we must ensure that there is some finite stage s where we select a path p through the i^{th} tree T, where we select an infinite set H_s homogeneous for p, and where we add H_s to $S(\mathcal{M})$ and close downward under $\leq_T$. To ensure that $\mathcal{M} \not\models \mathsf{WKL}_0$, we use Theorem 25 to select H_s s.t. $H_s \oplus \bigoplus_{j \leq s-1} H_j \not\geq \emptyset$.

It is possible that adding the set H_s to $S(\mathcal{M})$ causes new sets to become arithmetically-definable with parameters from $S(\mathcal{M})$. Therefore, each time we add H_s to $S(\mathcal{M})$, we create a new list containing the trees arithmetically-definable from $\bigoplus_{i \leq s} H_i$. We dovetail the lists, eventually running the general strategy for each tree in each list. In the limit, we obtain $\mathcal{M} \models \mathsf{RKL}^{(<\omega)} + \neg \mathsf{WKL}_0$. □

Corollary 27 (RCA_0)**.** $\mathsf{RKL}^{(<\omega)}$ *does not imply* WKL_0*.*

We separate $\mathsf{RKL}^{(<\omega)}$ from $\mathsf{RKL}^{(1)}$ with an ω-model by the following observation.

Lemma 28. *For each n, no model of* $\mathsf{RKL}^{(<\omega)}$ *is bounded by $\emptyset^n$.*

Proof. By the proof of lemma 7 relativized to $X = \emptyset^n$, we obtain an $\emptyset^n$-computable infinite tree T s.t. no infinite set $W_e^{\emptyset^n}$ is homogeneous for a path through T. Since each $\emptyset^n$-computable set is $W_e^{\emptyset^n}$ for some e, it follows that no infinite $\emptyset^n$-computable set is homogeneous for a path through T. □

Proposition 29. RT^2_2 *does not imply* $\mathsf{RKL}^{(<\omega)}$ *over* RCA_0. *Consequently,* $\mathsf{RKL}^{(1)}$ *does not imply* $\mathsf{RKL}^{(<\omega)}$ *over* RCA_0.

Proof. By the previous lemma, there is no model of $\mathsf{RKL}^{(<\omega)}$ which is bounded by $\emptyset^2$. By Theorem 3.1 of [1], there is an ω-model of RT^2_2 consisting of only low_2 sets. This model is bounded by $\emptyset^2$ so is not a model of $\mathsf{RKL}^{(<\omega)}$. □

Question 30. Does $\mathsf{RKL}^{(<\omega)}$ imply COH over RCA_0? Equivalently, does $\mathsf{RKL}^{(<\omega)}$ imply RT^2_2 over RCA_0?

4.1. **Subsets, co-subsets, and trees.** There is a close relationship between finding subsets/cosubsets of a fixed set, and finding sets that are homogeneous for a path through a fixed tree.

Statement 31 (RCA_0)**.** We define $D^{<\omega}_2$ to be the axiom scheme which asserts D^n_2 for each $n \in \omega$.

Proposition 32 (RCA_0)**.** $\mathsf{RKL}^{(<\omega)}$ *implies* $D^{<\omega}_2$.

Proof. Let $\mathcal{M} = (\mathbb{N}, S(\mathcal{M})) \models \mathsf{RCA}_0 + \mathsf{RKL}^{(<\omega)}$ and suppose that A is a Δ^0_n-definable subset of $\mathbb{N}$. We give a Π^0_n definition for a tree T as follows. Given $\tau \in 2^{<\mathbb{N}}$, we say that $\tau \in T$ if and only if $(\forall x < |\tau|)[\tau(x) = 1$ if and only if $x \in A]$.

By $\mathsf{RKL}^{(<\omega)}$, there is a set $H \in S(\mathcal{M})$ which is homogeneous for arbitrarily long strings in T with color $c \in \{0, 1\}$. Note that the only strings in T are initial segments of χ_A, so H is homogeneous for χ_A with color c. Then $H \subseteq A$ if $c = 1$, and $H \subseteq \overline{A}$ if $c = 0$, as desired. □

Remark 33. For ω-models, the reverse implication also holds.

Question 34. Does $D^{<\omega}_2$ imply $\mathsf{RKL}^{(<\omega)}$ over RCA_0?

By results of [1], SRT^2_2 implies $\mathrm{B}\Sigma^0_2$.

Question 35. Are there first order consequences of $\mathsf{RKL}^{(<\omega)}$ beyond $\mathrm{B}\Sigma^0_2$?

Chong, Slaman, and Yang have recently announced a proof that D^2_2 does not imply COH over RCA_0 [3].

Question 36. Does D^n_2 imply COH for any $n \in \omega$?

Theorem 2.1 of [9] gives another way to state this question for ω-models.

Question 37. Is there any arithmetically-definable $f : \mathbb{N} \to \{0, 1\}$ such that any set H homogeneous for f satisfies $H' \gg \emptyset'$?

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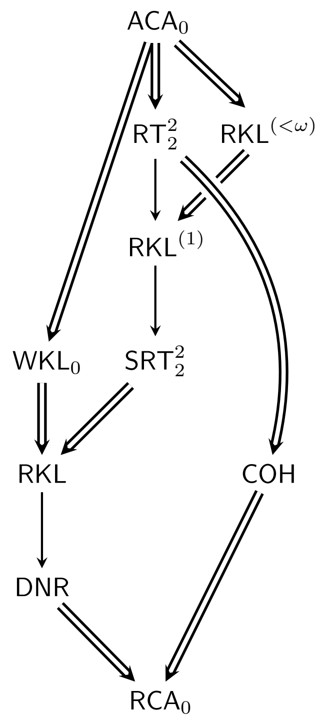

FIGURE 1. A summary of the principles considered.

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