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Echelon Form, Reduced Echelon Form, and Pivots

It helps to **circle** the *leftmost nonzero entries* (LNZE) of a matrix.

Define: A matrix is in *echelon form* (aka row echelon form) if
its *leftmost nonzero entries* move down and to the right, never skipping a row

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Echelon **Not** Echelon **Not** Echelon

Define: An echelon form matrix is in *reduced echelon form* if it also satisfies
1. all *LNZE* are 1
2. there are zero's *above* each LNZE

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Reduced Echelon **Not** Reduced **Not** Reduced

Matrices have *many* equivalent echelon forms, but
Theorem 1: Each matrix is equivalent to a **unique** *reduced* echelon matrix.

Note: *All* echelon forms of a matrix A have LNZE in the same place,
because reducing an echelon form matrix does *not* move LNZE.

A **Pivotal** Definition.

Define: Fix a matrix A.

- a *pivot position* is a leftmost nonzero entry in any **echelon** form of A, and
- a *pivot column* is a column of A that contains a pivot position.

- Each *pivot column* corresponds to a *bound variable.*
- Each *non-pivot column* corresponds to a *free variable.*

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & -7 \\ 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \text{corresponds to the system} \quad \begin{cases} x_1 \quad +x_3 \quad +x_5 = -7 \\ \quad x_2 +2x_3 \qquad = -3 \\ \qquad\qquad x_4+x_5 = 0 \end{cases}$$

E.g. x_1, x_2, x_4 are *bound* variables because there *are* pivots in columns 1, 2, and 4
and x_3, x_5 are *free* variables because there are are *not* pivots in columns 3 or 5.

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Solving A System: A Concrete Example

Step 1: Reduce the system to *echelon* form,
picking pivots starting from the *left* and clearing *down*.

Step 2: Further reduce the system to *reduced echelon* form,
picking pivots starting from the *bottom right* and clearing *up*.

Step 1: Reduce to *echelon* form and find pivot columns (aka leftmost variables)

$$\begin{cases} 2x_1 + x_2 + x_4 = 4 \\ x_1 + x_3 + x_4 = 0 \\ x_2 + x_4 = 2 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 4 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$\rho_1 \leftrightarrow \rho_2$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_4 = 4 \\ x_2 + x_4 = 2 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$\rho_2^\star = \rho_2 + (-2)\rho_1$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 - 2x_3 - x_4 = 4 \\ x_2 + x_4 = 2 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 4 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$\rho_3^\star = \rho_3 + (-1)\rho_2$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 - 2x_3 - x_4 = 4 \\ 2x_3 + 2x_4 = -2 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 4 \\ 0 & 0 & 2 & 2 & -2 \end{array}\right]$$

Step 2: Reduce to *reduced echelon* form

$\rho_3^\star = \frac{1}{2}\rho_3$

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 - 2x_3 - x_4 = 4 \\ x_3 + x_4 = -1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$\rho_1^\star = \rho_1 - \rho_3$

$\rho_2^\star = \rho_2 + 2\rho_3$

$$\begin{cases} x_1 = 1 \\ x_2 + x_4 = 2 \\ x_3 + x_4 = -1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

The variables x_1, x_2, x_3 are *bound* because there *are* pivots in columns 1,2,3.

The variable x_4 is *free* because there is *no* pivot in column 4.

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Solving A System: A General Algorithm

Step 1: Reduce the system to *echelon* form,
picking pivots starting from the *left* and clearing *down*.
Step 2: Further reduce the system to *reduced echelon* form,
picking pivots starting from the *bottom right* and clearing *up*.

Notation: $\star$ represents *any* number. $\blacksquare$ represents any *nonzero* number. 0 represents *zero*.

Step 1: Reduce to echelon form FAST (from LEFT to RIGHT)

1. (a) Start with the *leftmost non-zero column*. This is a pivot column.
   (b) Ensure the top position is a pivot by interchanging rows (if necessary).

$$\begin{bmatrix} 0 & \star & \star \\ \blacksquare & \star & \star \\ \blacksquare & \star & \star \\ 0 & \star & \star \end{bmatrix} \underset{\rho_1 \leftrightarrow \rho_3}{\sim} \begin{bmatrix} \blacksquare & \star & \star \\ \blacksquare & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{bmatrix}$$

2. Use Replacement to create zero's in all positions *below* the pivot.

$$\underset{\rho_2^* = \rho_2 + (\#) \cdot \rho_1}{\sim} \begin{bmatrix} \blacksquare & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{bmatrix}$$

3. Repeat steps 1-3 for the circled submatrix. Keep going until the matrix is in echelon form.

   For example, if the matrix in (2) is $\begin{bmatrix} \blacksquare & \star & \star \\ 0 & 0 & \blacksquare \\ 0 & \blacksquare & \star \\ 0 & \blacksquare & \star \end{bmatrix}$

$$\underset{\rho_2 \leftrightarrow \rho_4}{\sim} \begin{bmatrix} \blacksquare & \star & \star \\ 0 & \blacksquare & \star \\ 0 & \blacksquare & \star \\ 0 & 0 & \blacksquare \end{bmatrix} \underset{\rho_3^* = \rho_3 + (\#) \cdot \rho_2}{\sim} \begin{bmatrix} \blacksquare & \star & \star \\ 0 & \blacksquare & \star \\ 0 & 0 & \star \\ 0 & 0 & \blacksquare \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \blacksquare & \star & \star \\ 0 & \blacksquare & \star \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2: Clear out the pivot columns (from BOTTOM to TOP)

1. Starting with *rightmost pivot*, and working up and to the left, create zeros above each pivot. If the pivot is not 1, rescale the row to make it 1.

$$\underset{\rho_2^* = \rho_2 + (\#) \cdot \rho_3}{\sim} \begin{bmatrix} \blacksquare & \star & \star \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix} \underset{\rho_1^* = \rho_1 + (\#) \cdot \rho_3}{\sim} \begin{bmatrix} \blacksquare & \star & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\rho_1^* = \rho_1 + (\#) \cdot \rho_2}{\sim} \begin{bmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$