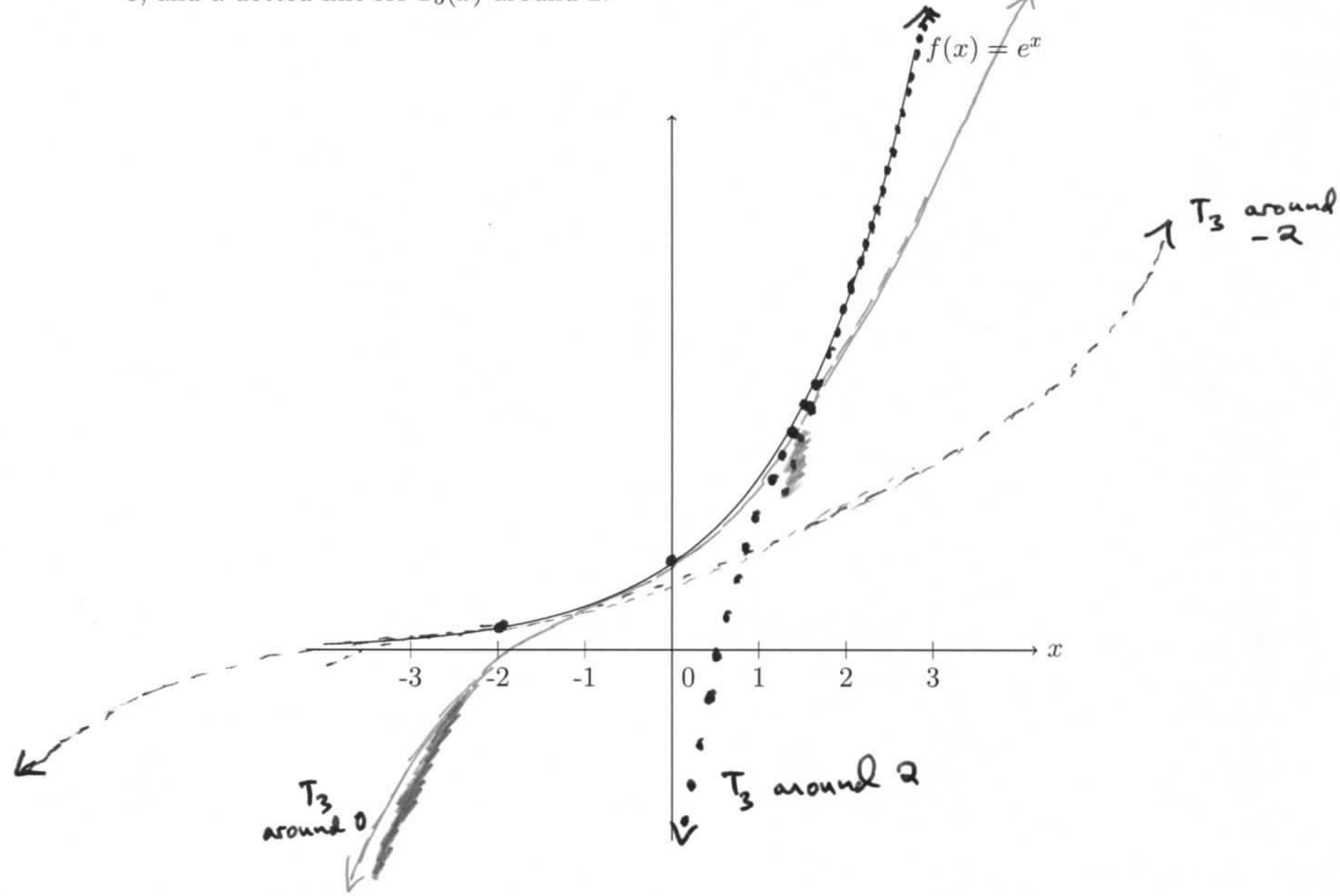
Written Homework #5. Math 141 Due April 10, 2013

Name: Key Section: ______

1. First, write down the third Taylor polynomials of $y = e^x$ centered at -2, at 0, and at 2.

   Then, sketch them below. Use short dashes for $T_3(x)$ around -2, a solid line for $T_3(x)$ around 0, and a dotted line for $T_3(x)$ around 2.



Remember that around a

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

and that for all n, $f^{(n)}(x) = e^x$

Therefore

around -2:

$$T_3(x) = e^{-2} + e^{-2}(x+2) + \frac{e^{-2}}{2\cdot 1}(x+2)^2 + \frac{e^{-2}}{3\cdot 2\cdot 1}(x+2)^3$$

around 0:

$$T_3(x) = 1 + 1\cdot x + \frac{1}{2\cdot 1}x^2 + \frac{1}{3\cdot 2\cdot 1}x^3$$

around 2:

$$T_3(x) = e^2 + e^2(x-2) + \frac{e^2}{2\cdot 1}(x-2)^2 + \frac{e^2}{3\cdot 2\cdot 1}(x-2)^3$$

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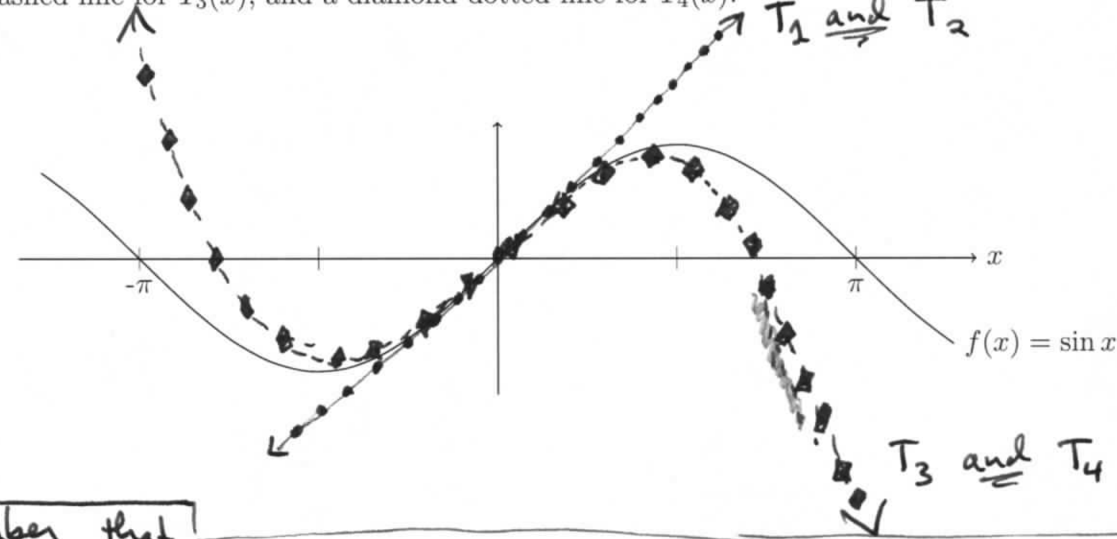
2. First, write down the Taylor polynomials T_1, T_2, T_3, and T_4 of $y = \sin(x)$ all centered at $a = 0$.

Then, sketch them on the graph below. Use a solid line for $T_1(x)$, a dotted line for $T_2(x)$, a dashed line for $T_3(x)$, and a diamond-dotted line for $T_4(x)$.



Remember that around 0, $T_n(x) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^3}{2!} + \cdots + \frac{f^{(n)}(0)}{n!}\cdot x^n$

[redacted]

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0

$$T_1(x) = f(0) + \frac{f'(0)}{1!}x = 0 + 1\cdot x = x$$

$$T_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 0 + x + 0\cdot x^2 = x$$

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + x + 0\cdot x^2 + \frac{-1}{3!}x^3$$
$$= x - \frac{x^3}{6}$$

$$T_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 0 + x + 0 - \frac{x^3}{6} + 0\cdot x^4 = x - \frac{x^3}{6}$$

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3. Find the value of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$. $= e^{2}$

Remember: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. Find the value of the series $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$. Simplify completely.

Remember: $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

So $\sum_{n=0}^{\infty} (-1) \frac{\pi^{2n}}{(2n)!} = \cos(\pi) = -1$

5. Use the Taylor error estimate to determine the maximum error of degree the 4^{th} degree Taylor polynomial of $\sin(x)$ centered around 0 on the interval $[-\pi, \pi]$.

Using your calculator, find a decimal value for this error.

[redacted]

① find M

[redacted]

check: $\frac{d^5}{dx^5} \sin(x) = \cos(x)$

notice: $|\cos(x)| \leq 1$ for all x in $[-\pi, \pi]$

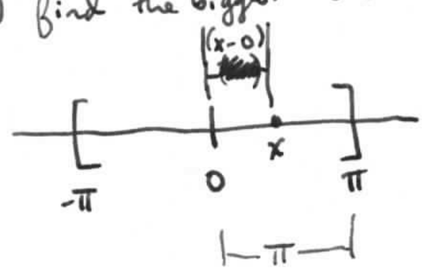
set: $M = 1$

Remember:

If $f^{(4+1)}(x) \leq$ some # M for all x in $[-\pi, \pi]$

Then $\left| \begin{matrix} \text{error of} \\ T_4(x) \end{matrix} \right| = |R_4(x)| \leq \frac{M}{(4+1)!} (x-0)^{4+1}$ for all x in $[-\pi, \pi]$

② find the biggest distance $x - 0$



③ conclude

$\left| \begin{matrix} \text{error of} \\ T_4 \text{ on} \\ [\pi, \pi] \end{matrix} \right| \leq \frac{1}{5!} (\pi)^5 = \frac{\pi^5}{5!} \approx 2.55$

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Optional problems for further study

6. (**Optional**) What is the smallest degree Taylor polynomial of $\sin(x)$ centered around 0 required to approximate $\sin(x)$ on $[-\pi, \pi]$ with error less than or equal to $\frac{\pi^{11}}{11!} \approx 0.0074$.

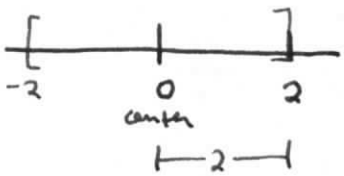
for <u>all</u> n, $\left|\frac{d^{n+1}}{dx^{n+1}} \sin(x)\right| \leq 1$ for all x in $[-\pi, \pi]$

<u>and</u> π is the greatest distance from the center 0

<u>so:</u> $\left|\begin{matrix}\text{error of} \\ T_n(x) \\ \text{on } [-\pi,\pi]\end{matrix}\right| \leq \frac{1}{(n+1)!}\pi^{n+1} = \frac{\pi^{n+1}}{(n+1)!}$ for all n

so $T_{10}(x)$ is required to approximate $\sin(x)$ on $[-\pi, \pi]$ with error $\leq \frac{\pi^{11}}{11!}$

7. (**Optional**) What is the smallest degree Taylor polynomial of e^x centered around 0 required to approximate e^2 with error less than or equal to $\frac{e^2 \cdot 2^{11}}{(11)!}$. Using your calculator, find a decimal value for this error.



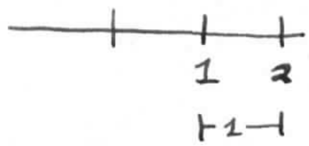
for all n, $|f^{(n+1)}(x)| \leq e^2$ for all x in $[-2, 2]$

and $2 - 0 = 2$

so $\left|\begin{matrix}\text{error of} \\ T_n(x)\end{matrix}\right| \leq \frac{e^2}{(n+1)!}(2-0)^{n+1} = \frac{e^2 \cdot 2^{n+1}}{(n+1)!}$.

so error of $T_{10}(2)$ is at most $\frac{e^2 \cdot 2^{11}}{11!} \approx .000379$

8. (**Optional**) What is the smallest degree Taylor polynomial of e^x centered around 1 required to approximate e^2 with error less than or equal to $\frac{e^2}{(11)!}$. Using your calculator, find a decimal value for this error.



for all n, $|f^{(n+1)}(x)| \leq e^2$ for all x in $[-2, 2]$

and $(2-1) = 1$

so $\left|\begin{matrix}\text{error of} \\ T_n(x)\end{matrix}\right| \leq \frac{e^2}{(n+1)!}(2-1)^{n+1} = \frac{e^2}{(n+1)!}$

So error of $T_{10}(2)$ is at most $\frac{e^2}{11!} \approx .000\,000\,185$