Written Homework #3. Math 141 Due March 22, 2013

Name: Key Section: ______________

1. Let $\{a_n\}$ be a mystery sequence, with $\sum_{n=1}^{\infty} a_n = \pi$. Compute $\lim_{n\to\infty} a_n$.

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If $\lim_{n\to\infty} a_n \neq 0$

Then $\sum_1^{\infty} a_n$ diverges

Because $\sum_1^{\infty} a_n$ converges

$\boxed{\lim_{n\to\infty} a_n = 0}$

1pt

- Knowing relative rates of growth can help you compute certain limits quickly.

  Recall that $f(x) \ll g(x)$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
  The following rates of growth are worth knowing well (memorizing):

$$\ln(x) \ll x \ll x^2 \ll (\sqrt{2})^x \ll 2^x \ll e^x \ll 3^x \ll \pi^x \ll 4^x \ll x^x$$

  We said that f grows at the same rate as g, written $f \asymp g$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = C \neq 0$ is a finite number. For example,

$$x \asymp 2x + 1$$

$$x^2 \asymp 5x^2 - 3x + 1$$

- Later, we showed that for for sequences, $n^n \gg n!$. In other words,

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n^n}{n!} = \infty$$

  Similarly, we showed that $n! \gg a^n$ for any fixed finite number $a \neq 0$. In other words,

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n!}{a^n} = \infty$$

  Use these facts to help you certain compute limits quickly.

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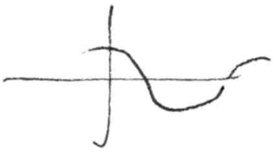
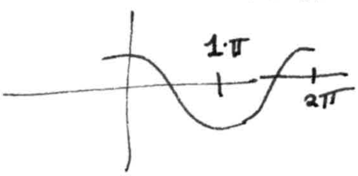
2. Determine if the following **sequences** converge or diverge. If it converges, find the limit. If it diverges, circle the answer.

(a) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\} = \{1, 0, -1, 0, 1, 0, \ldots\}$



Converges to ______________________ diverges 1 pt

(b) $\{(-1)^n + \cos(n\pi)\} = \{(-1)+(-1), (-1)^2 + 1, \ldots\}$

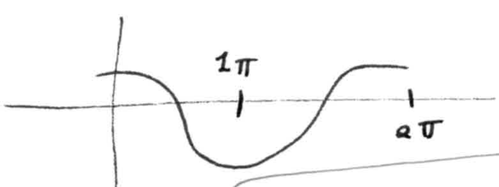


$= \{-2, 2, -2, \ldots\}$

Converges to ______________________ diverges 1 pt

(c) $\{(-1)^{n-1} + \cos(n\pi)\} = \{(-1)^{1-1} + (-1), (-1)^{2-1} + 1, (-1)^{3-1} + (-1), \ldots\}$

$= \{1 + -1, -1 + 1, 1 + -1, \ldots\}$



$= \{0, 0, 0, \ldots\}$

$\boxed{\text{Converges to } 0}$ 1 pt diverges

(d) $\left\{\frac{4^n}{(-3)^{2n}}\right\} = \left\{\frac{4^n}{9^n}\right\} = \left\{\left(\frac{4}{9}\right)^n\right\}$

and $\lim_{n\to\infty} \left(\frac{4}{9}\right)^n = 0$

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$\boxed{\text{Converges to } 0}$ 1 pt diverges

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3. For each **sequence** defined below, does $\{a_n\}$ converge? If so, what does it converge to? If it diverges, circle the answer.

(a) Let $a_n = \sqrt{\frac{2n-1}{2n+2}}$. $= \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}} \cdot \sqrt{\frac{2-\frac{1}{n}}{2+\frac{2}{n}}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{2-\frac{1}{n}}{2+\frac{2}{n}}} = \sqrt{\frac{2}{2}} = 1$$

$\frac{1}{n} \to 0$, $\frac{2}{n} \to 0$

$\boxed{\text{Converges to } 1}$ diverges

(b) Let $a_n = \frac{2+e^{-n+1}-e^{-n}}{\sqrt{n}}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+e^{-n+1}-e^{-n}}{\sqrt{n}} = 0$$

$-n+1 \to -\infty$, $e^{-n+1} \to 0$, $e^{-n} \to 0$, $\sqrt{n} \to \infty$

$\boxed{\text{Converges to } 0}$ diverges

(c) Let $a_n = \frac{e^{-n}}{\sin(\frac{1}{n})}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{-n}}{\sin(\frac{1}{n})} \stackrel{H}{=} \lim_{n\to\infty} \frac{-e^{-n}}{\cos(\frac{1}{n})\cdot\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{-e^{-n}}{\frac{-1}{n^2}}$$

$e^{-n} \to 0$, $\sin(\frac{1}{n}) \to 0$, $\frac{1}{n} \to 0$, $\cos(\frac{1}{n}) \to 1$

$$= \lim_{n\to\infty} \frac{n^2}{e^n} = 0$$

(by relative rates of growth)

$\boxed{\text{Converges to } 0}$ diverges

(d) Let $a_n = \frac{n\cos(n)}{n^2+1}$. $= \frac{\cancel{n}}{n^{\cancel{2}}} \cdot \left(\frac{\cos(n)}{1+\frac{1}{n^2}}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\text{goes to } 0} \cdot \underbrace{\left(\frac{\cos(n)}{1+\frac{1}{n^2}}\right)}_{\text{oscillates between bounded \#'s}} = 0$$

by squeeze theorem.

$\boxed{\text{Converges to } 0}$ diverges

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4. Determine if the following **sequences** converge or diverge. If it converges, find the limit. If it diverges, circle the answer.

(a) $\left\{ \dfrac{\sin^3(n)}{n^3} \right\}$ $\sin^3(n)$ bounded betwe

$-1 \le \sin^3(n) \le 1$

$\Rightarrow \dfrac{-1}{n^3} \le \dfrac{\sin^3(n)}{n^3} \le \dfrac{1}{n^3} \to 0$

$\boxed{\text{Converges to } 0}$ 1pt ~~diverges~~

(b) $\left\{ \dfrac{(-1)^{n-1} \cdot n!}{5^n + \cos(n)} \right\}$ $=$ $a_n = \dfrac{n!\,(-1)^{n-1}}{5^n\left(1 + \dfrac{\cos(n)}{5^n}\right)}$

$\boxed{\text{Remember: } n! \gg 5^n}$ $\Rightarrow$ goes to ∞

oscillating betwo finite #'s

Converges to ______________________ $\boxed{\text{diverges}}$ 1pt

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5. For the following alternating sums, find the least number of terms required to get

$|error| \leq .00000001 = \frac{1}{10^8}$

alternating series error estimate

$\left| \begin{matrix} \text{error of} \\ \text{sum of first} \\ n \text{ terms} \end{matrix} \right| \leq b_{n+1}$

positive part of $n+1^{st}$ term

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

$\left| \begin{matrix} \text{error of} \\ S_n \end{matrix} \right| \leq b_{n+1} = \frac{1}{(n+1)^4} \leq \frac{1}{10^8}$

if $(n+1)^4 \geq 10^8$

$n+1 \geq 10^2$

$n \geq 99$

$\boxed{99^{th} \text{ term is least}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$\left| \begin{matrix} \text{error of} \\ S_n \end{matrix} \right| \leq b_{n+1} = \frac{1}{n+1} \leq \frac{1}{10^8}$

100000000.00

if $n+1 \geq 10^8$

$n \geq 99{,}999{,}999$ ← this is least # of terms

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^n}$

$\left| \begin{matrix} \text{error of} \\ S_n \end{matrix} \right| \leq b_{n+1} = \frac{1}{10^{n+1}} \leq \frac{1}{10^8}$

if $n+1 \geq 8$

$n \geq 7$ ← (this is least # of terms)

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6. Suppose we know that the **series** $\sum_{n=1}^{\infty} a_n = \frac{e^2}{\pi}$. Compute the following **sequence** limits.

(a) $\lim_{n\to\infty} 2 \cdot a_n + 5$

$= 2 \cdot 0 + 5 = 5$

If $\lim_{n\to\infty} a_n \neq 0$

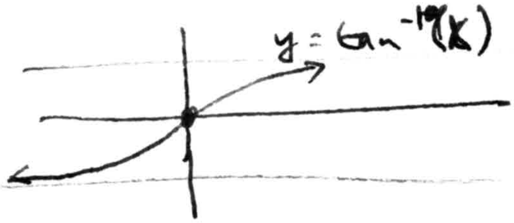
Then $\sum a_n$ diverges

So Because $\sum a_n$ converges

$\lim_{n\to\infty} a_n = 0$

(b) $\lim_{n\to\infty} a_n \cdot \boxed{\sin(a_n)} = 0$

(c) $\lim_{n\to\infty} \boxed{\tan^{-1}(2\pi \cdot a_n)} = 0$



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