Review Sheet #3 Math 141 Integrals and L'Hospital's Rule

Name: ______________________ Section: ______________________

1. *Simplify if possible:* Algebra

Evaluate the integral $\int x \cdot (1+x)^2\,dx$. $= \int x\,(1+2x+x^2)\,dx$

$= \int [x + 2x^2 + x^3]\,dx$

$= \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + C$

2. *Simplify if possible:* Trigonometry

Evaluate the integral $\int \frac{\tan(x)}{\sec^2(x)}\,dx$. $= \int \frac{\frac{\sin x}{\cos x}}{\left(\frac{1}{\cos x}\right)^2}\,dx$

$= \int \sin x \cdot \cos x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$

$\begin{pmatrix} u = \sin x \\ du = \cos x\,dx \end{pmatrix}$

3. *Simplify if possible:* A mix of both

Evaluate the integral $\int (\sin(x) + \cos(x))^2\,dx$. $= \int [\sin^2 x + 2\sin x \cos x + \cos^2 x]\,dx$

(Know: $\sin^2 x + \cos^2 x = 1$)

$= \int [1 + 2\sin x \cos x]\,dx = x + 2\int \sin x \cos x\,dx = x + 2\frac{\sin^2 x}{2} + C$

$\begin{pmatrix} u = \sin x \\ du = \cos x \end{pmatrix}$ $= x + \sin^2 x + C$

4. *Look for an obvious substitution*

Evaluate the integral $\int x\,e^{-x^2}\,dx$.

$\begin{pmatrix} u = -x^2 \\ du = -2x\,dx \\ -\frac{du}{2} = x\,dx \end{pmatrix}$

$= \int e^u \cdot \frac{-du}{2}$

$= -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$

5. *Look for an obvious substitution*

Evaluate the integral $\int \frac{1}{x \cdot (\ln x)^3}\,dx$.

$\begin{pmatrix} u = \ln(x) \\ du = \frac{1}{x}\,dx \end{pmatrix}$

$= \int \frac{1}{u^3}\,du = \int u^{-3}\,du = \frac{u^{-2}}{-2} + C$ [redacted]

$= \frac{-1}{2u^2} + C = \frac{-1}{2(\ln(x))^2} + C$

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6. *Classify according the integral's form:* Trigonometric: odd power of $\cos(x)$ or $\sin(x)$

Evaluate the integral $\int (\sin^3(x) + \sin(x)\cos(x))\, dx$.

Remember:
$$\sin^2 x + \cos^2 x = 1$$
$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$= \int \left(\sin^2(x) + \cos(x)\right) \cdot \sin(x)\, dx$$

$$= \int \left(1 - \cos^2 x + \cos x\right) \cdot \sin x\, dx \qquad \begin{pmatrix} u = \cos x \\ du = -\sin x\, dx \end{pmatrix}$$

$$= \int -(1 - u^2 + u)\, du = -u + \frac{u^3}{3} - \frac{u^2}{2} + C = \boxed{-\cos x + \frac{\cos^3(x)}{3} - \frac{\cos^2(x)}{2} + C}$$

7. *Classify according the integral's form:* Trigonometric: even power of $\cos(x)$ or $\sin(x)$

Evaluate the integral $\int (\cos^2(x) - \sin^2(x))\, dx$.

Remember:
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

[redacted]

$$= \int \left[\frac{1 + \cos(2x)}{2} - \frac{1 - \cos(2x)}{2}\right] dx$$

$$\begin{pmatrix} u = 2x \\ du = 2dx \\ \frac{du}{2} = dx \end{pmatrix}$$

$$= \int \cos(2x)\, dx = \boxed{\frac{1}{2}\sin(2x) + C}$$

8. *Classify according the integral's form:* Trigonometric: a nice product of $\tan(x)$ and $\sec(x)$.

Evaluate the integral $\int \tan^3(x)\sec(x)\, dx$.

Remember: $\frac{d}{dx}\sec x = \tan(x)\sec(x)$

Remember: $\tan^2 x + 1 = \sec^2 x$
$\Rightarrow \tan^2 x = \sec^2 x - 1$

$$= \int \tan^2(x) \cdot \tan(x)\sec(x)\, dx$$

$$= \int (\sec^2 x - 1) \cdot \tan(x)\sec(x)\, dx$$

$u = \sec(x) \Rightarrow du = \tan x \sec x\, dx$

$$= \int (u^2 - 1)\, du = \frac{u^3}{3} - u + C = \boxed{\frac{\sec^3(x)}{3} - \sec(x) + C}$$

9. *Classify according the integral's form:* Trigonometric: a nice product of $\tan(x)$ and $\sec(x)$.

Evaluate the integral $\int \tan(x)\sec^2(x)\, dx$.

Remember: $\frac{d}{dx}\tan(x) = \sec^2(x)$

[redacted]

$$\begin{pmatrix} u = \tan x \\ \Rightarrow du = \sec^2 dx \end{pmatrix}$$

$$= \int u\, du = \frac{u^2}{2} + C = \boxed{\frac{\tan^2(x)}{2} + C}$$

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10. *Classify according the integral's form:* Rational Polynomial:
Proper, 2 distinct linear polynomials.

Evaluate the integral $\int \frac{1}{(x+1)(x-1)}\,dx$.

$$= \int \left(\frac{-1}{2} \frac{1}{(x+1)} + \frac{1}{2} \frac{1}{(x-1)} \right) dx$$

$$= \frac{-1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C$$

this is done integrating.

But we can still simplify it more!

$$= \ln \sqrt{|x-1|} - \ln \sqrt{|x+1|} + C$$

$$= \ln \sqrt{\left| \frac{x-1}{x+1} \right|} + C$$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

so

$$1 = A(x-1) + B(x+1)$$

$x = 1 \Rightarrow 1 = 0 + 2B$

$\Rightarrow B = \frac{1}{2}$

$x = -1 \Rightarrow 1 = -2A + 0$

$\Rightarrow A = -\frac{1}{2}$

11. *Classify according the integral's form:* Rational Polynomial:
Proper, several distinct & repeated linear polynomials.

Evaluate the integral $\int \frac{4x+1}{(x-2)(x+1)^2}\,dx$.

$$= \int \left(\frac{1}{x-2} + \frac{-1}{x+1} + \frac{1}{(x+1)^2} \right) dx$$

$$= \ln|x-2| - \ln|x+1| + \int \frac{1}{(x+1)^2}\,dx$$

$$\begin{pmatrix} u = x+1 \\ du = dx \end{pmatrix}$$

$$= \ln\left| \frac{x-2}{x+1} \right| + \underbrace{\int u^{-2}\,du}_{= \frac{-1}{u} + C}$$

$$= \ln\left| \frac{x-2}{x+1} \right| + \frac{-1}{x+1} + C$$

$$\frac{4x+1}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

so

$$4x+1 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

$x = -1 \Rightarrow -3 = 0 + 0 + -3 \cdot C$

$\Rightarrow C = 1$

$x = 2 \Rightarrow 9 = 9 \cdot A + 0 + 0$

$\Rightarrow A = 1$

(can't kill off A&C while keeping B. Just plug in a nice #)

$x = 0 \Rightarrow 1 = A + (-2)B + (-2)C$

$1 = 1 + -2B + -2$

$B = -1$

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12. *Classify according the integral's form:* Rational Polynomial:
Improper, distinct linear polynomials.

Evaluate the integral $\int \frac{x^2}{x^2-3x+2}\,dx$.

NOT a proper fraction, so can't use partial fractions yet

$$x^2-3x+2 \;\overline{)\; x^2 + \_\_ + \_\_ \;}\quad 1$$
$$\underline{-(x^2-3x+2)}$$
$$+3x-2$$

$$= \int \left[1 + \frac{3x-2}{x^2-3x+2} \right] dx$$

$$= x + \int \frac{3x-2}{(x-1)(x-2)}\,dx$$

$$\frac{3x-2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

so

$$3x-2 = A(x-2) + B(x-1)$$
$$x=2 \Rightarrow 4 = 0 + B \Rightarrow B = 4$$
$$x=1 \Rightarrow 1 = (-1)A + 0 \Rightarrow A = -1$$

$$= x + \int \left[\frac{-1}{x-1} + \frac{4}{x-2} \right] dx$$

$$= x - \ln|x-1| + \ln|x-2| + C$$

$$\boxed{= x + \ln\left|\frac{x-2}{x-1}\right| + C}$$

13. *Classify according the integral's form:* Rational Polynomial:
Proper, distinct irreducible quadratic.

Evaluate the integral $\int \frac{x-1}{x(x^2+1)}\,dx$.

$$\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$$

so

$$x-1 = A(x^2+1) + x(Bx+C)$$
$$x-1 = Ax^2 + A + Bx^2 + Cx$$

collecting coefficients, we get

$$0\cdot x^2 + 1\cdot x + (-1) = (A+B)\cdot x^2 + C\cdot x + A$$

So: $A = -1$

$C = 1$

$A + B = 0 \Rightarrow B = 1$

$$= \int \left[\frac{-1}{x} + \frac{1\cdot x + 1}{(x^2+1)} \right] dx$$

$$= -\ln|x| + \int \frac{x}{(x^2+1)}\,dx + \int \frac{1}{(x^2+1)}\,dx$$

$$\begin{pmatrix} u = x^2+1 \\ du = 2x\,dx \\ \frac{du}{2} = x\,dx \end{pmatrix} \qquad \int \frac{1}{(x^2+1)}\,dx = \tan^{-1}(x)$$

$$= -\ln|x| + \int \frac{1}{2}\frac{1}{u}\,du + \tan^{-1}(x) + C$$

$$\boxed{= -\ln|x| + \frac{1}{2}\cdot\ln|x^2+1| + \tan^{-1}(x) + C}$$

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14. *Classify according the integral's form:* Trig Substitution: $x = \sin(\theta)$, indefinite.

Evaluate the integral $\int \frac{1}{\sqrt{25-9x^2}}\,dx. = \int \frac{1}{5} \frac{1}{\sqrt{1-\left(\frac{3x}{5}\right)^2}}\,dx$

Remember
$\cos^2\theta = 1 - \sin^2\theta$

set: $\frac{3x}{5} = \sin\theta$

$\frac{3}{5}dx = \cos\theta\, d\theta$

$dx = \frac{5}{3}\cos\theta\, d\theta$

$\frac{3x}{5} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}\left(\frac{3x}{5}\right)$

$= \int \frac{1}{5} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{5}{3}\cos\theta\, d\theta$

$= \int \frac{1}{5}\cdot\frac{5}{3}\cdot\frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{1}{3}\,d\theta = \frac{\theta}{3} + C = \frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right) + C$

15. *Classify according the integral's form:* Trig Substitution: $x = \sec(\theta)$, indefinite.

Evaluate the integral $\int \frac{1}{x\sqrt{x^2-9}}\,dx. = \int \frac{1}{x\cdot 3\cdot\sqrt{\left(\frac{x}{3}\right)^2-1}}\,dx$

Remember
$\tan^2\theta = \sec^2\theta - 1$

$\frac{x}{3} = \sec\theta$

$\boxed{x = 3\cdot\sec\theta}$

$\frac{dx}{3} = \sec\theta\cdot\tan\theta\cdot d\theta$

$dx = 3\cdot\sec\theta\tan\theta\, d\theta$

$\frac{x}{3} = \sec\theta$

$\Rightarrow \theta = \sec^{-1}\left(\frac{x}{3}\right)$

$= \int \frac{1}{\boxed{3\cdot\sec\theta}\, 3\sqrt{\sec^2\theta - 1}} \cdot 3\cdot\sec\theta\tan\theta\, d\theta$

$= \int \frac{3\cdot\sec\theta\tan\theta}{3\cdot 3\cdot\sec\theta\tan\theta}\,d\theta = \int \frac{1}{3}\,d\theta = \frac{\theta}{3} + C = \frac{\sec^{-1}\left(\frac{x}{3}\right)}{3} + C$

16. *Classify according the integral's form:* Trig Substitution: $x = \tan(\theta)$, indefinite.

Evaluate the integral $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx. = \int \frac{1}{x^2\cdot 2\cdot\sqrt{\left(\frac{x}{2}\right)^2+1}}\,dx$

Remember
$\sec^2\theta = \tan^2\theta + 1$

$\frac{x}{2} = \tan\theta \Rightarrow x = 2\tan\theta$

& $dx = 2\sec^2\theta\, d\theta$

$= \int \frac{1}{(2\tan\theta)^2\cdot 2\cdot\sqrt{\tan^2\theta+1}} \cdot 2\sec^2\theta\, d\theta$

$= \int \frac{2\cdot\sec^2\theta}{4\cdot\tan^2\theta\cdot 2\sqrt{\sec^2\theta}}\,d\theta = \int \frac{1}{4}\frac{\sec\theta}{\tan^2\theta}\,d\theta$

$\int \frac{1}{4}\frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int \frac{1}{4}\frac{\cos\theta}{\sin^2\theta}\,d\theta$

($u = \sin\theta$, $du = \cos\theta$)

$= \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\cdot\frac{-1}{u} + C$

$= \frac{1}{4}\cdot\frac{-1}{\sin\theta} + C$

$= \frac{-1}{4}\cdot\frac{1}{\frac{x}{\sqrt{x^2+4}}} + C$

$\frac{x}{2} = \tan\theta = \frac{opp}{adj}$

$= \frac{-\sqrt{x^2+4}}{4x} + C$

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17. *Classify according the integral's form:* Integration by parts: du is simpler, v is not worse

Evaluate the integral $\int \underbrace{\boxed{\ln(x)}}_{u} \underbrace{dx.}_{dv}$

$$\left(\begin{array}{ll} u = \ln(x) & dv = dx \\ du = \frac{1}{x}\,dx & v = x \end{array} \right)$$

$$\left(\int u\,dv = uv - \int v\,du \right)$$

$$= x \cdot \ln(x) - \int x \cdot \frac{1}{x}\,dx$$

$$= x \cdot \ln(x) - x + C$$

18. *Classify according the integral's form:* Integration by parts: du is simpler, v is not worse

Evaluate the integral $\int \underbrace{\boxed{\tan^{-1}(3x)}}_{u} \underbrace{dx.}_{dv}$

$$\left(\begin{array}{ll} u = \tan^{-1}(3x) & dv = dx \\ du = \frac{1}{1+9x^2} \cdot 3 & v = x \end{array} \right)$$

$$= x \cdot \tan^{-1}(3x) - \int \frac{3}{1+9x^2}\, x\,dx$$

$$\left(\begin{array}{l} u = 1 + 9x^2 \\ du = 18x\,dx \\ \frac{du}{6} = 3x\,dx \end{array} \right)$$

$$= x \cdot \tan^{-1}(3x) - \int \frac{1}{6u}\,du = x \tan^{-1}(3x) - \frac{1}{6}\ln|u| + C$$

$$\boxed{= x \cdot \tan^{-1}(3x) - \frac{1}{6}\ln\left|1 + 9x^2\right| + C}$$

19. *Classify according the integral's form:* Integration by parts: loops around

Evaluate the integral $\int \underbrace{e^x}_{dv}\, \underbrace{\cos(x)}_{u}\, dx$.

$$\left(\begin{array}{ll} u = \cos x & dv = e^x\,dx \\ du = -\sin x\,dx & v = e^x \end{array} \right)$$

$$= e^x \cdot \cos(x) - \int \underbrace{\boxed{-\sin x}}_{u}\, \underbrace{\boxed{e^x\,dx}}_{dv}$$

$$\left(\begin{array}{ll} u = \sin x & dv = e^x \\ du = \cos x & v = e^x \end{array} \right)$$

$$\int e^x \cos(x)\,dx = e^x \cdot \cos(x) + \left[e^x \cdot \sin(x) - \int \cos x\, e^x\,dx \right] + C$$

$\Rightarrow$ solving for $\int \cos x\, e^x\,dx$ we get

$$\boxed{\int \cos x\, e^x\,dx = \frac{1}{2}\left[e^x \cos x + e^x \sin x \right] + C}$$

20. If nothing has worked then *return to step 1*, "simplify if possible," and try again!

Look for clever tricks. Pay special attention to other ways of re-writing the problem, clever u-substitutions, clever inverse-substitutions, and clever uses of integration by parts.

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21. *L'Hopital's for limits of type $\frac{0}{0}$*

Does the limit $\lim\limits_{x\to 0} \dfrac{\tan^{-1}(x)}{x^3+6x}$ converge? If so, what does it converge to?

$\tan^{-1}(x) \to 0$, $x^3+6x \to 0$

$$\stackrel{H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{3x^2+6} = \lim_{x\to 0} \frac{1}{(1+\underbrace{x^2}_{\to 0})(\underbrace{3x^2}_{\to 0}+6)} = \frac{1}{6}$$

the limit converges to $\frac{1}{6}$

22. *L'Hopital's for limits of type $\frac{\infty}{\infty}$*

Does the limit $\lim\limits_{x\to 0+} \dfrac{\ln(\sin(3x))}{\ln(\tan(3x))}$ converge? If so, what does it converge to?

$\ln(\sin(3x)) \to -\infty$, $\ln(\tan(3x)) \to -\infty$

(as $x \to 0$, $\sin(3x) \to 0$, $\tan(3x) \to 0$

so as $x \to 0$, $\ln(\sin(3x)) \to -\infty$, $\ln(\tan(3x)) \to -\infty$)

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin(3x)}\cdot\cos(3x)\cdot\cancel{3}}{\frac{1}{\tan(3x)}\cdot\sec^2(3x)\cdot\cancel{3}}$$

$$= \lim_{x\to 0^+} \frac{\frac{\cancel{\cos(3x)}}{\cancel{\sin(3x)}}}{\frac{\cancel{\cos(3x)}}{\cancel{\sin(3x)}}\cdot\frac{1}{\cos^2(3x)}}$$

$$= \lim_{x\to 0^+} \cos^2(3x) = 1$$

23. *L'Hopital's for limits of type $0\cdot\infty$*

Does the limit $\lim\limits_{x\to\infty} \underbrace{2x}_{\infty} \underbrace{\sin\left(\frac{1}{2x}\right)}_{\to 0}$ converge? If so, what does it converge to?

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{2x}\right)}{\frac{1}{2x}} \;\begin{matrix}\to 0\\ \to 0\end{matrix} \stackrel{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{2x}\right)\cdot\frac{\cancel{-1}}{\cancel{2x^2}}}{\frac{\cancel{-1}}{\cancel{2x^2}}} = \lim_{x\to\infty} \underbrace{\cos\left(\overbrace{\frac{1}{2x}}^{\to 0}\right)}_{\to 1}$$

$$= 1$$

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24. *L'Hopital's for limits of type* $\infty - \infty$

Does the limit $\lim_{x\to\infty} [\underbrace{\ln(\underbrace{3x+2}_{\to\infty})}_{\to\infty} - \underbrace{\ln(\underbrace{2x+1}_{\to\infty})}_{\to\infty}]$ converge? If so, what does it converge to?

$$= \lim_{x\to\infty}\left(\ln\left(\frac{3x+2}{2x+1}\right)\right).$$

side work: as $x \to \infty$

$$\lim_{x\to\infty} \frac{3x+2}{2x+1} = \frac{3}{2}$$

so $\ln\left(\frac{3x+2}{2x+1}\right) = \ln\left(\frac{3}{2}\right)$

$$= \ln\left(\frac{3}{2}\right)$$

the limit does converge! to $\ln(\frac{3}{2})$.

25. *L'Hopital's for limits of type* 0^0

Does the limit $\lim_{x\to\infty} \left(\frac{4}{x}\right)^{\frac{1}{x}}$ converge? If so, what does it converge to?

$$\lim_{x\to\infty}\left(\frac{4}{x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\cdot\ln\left(\frac{4}{x}\right)}$$

(By definition of general exponential function)

$$= e^0 = 1$$

The limit does converge! to 1.

Sub-question: What is $\lim_{x\to\infty} \frac{1}{x}\cdot\ln\left(\frac{4}{x}\right)$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{4}{x}\right)}{x} \quad \begin{matrix}\nearrow -\infty \\ \searrow \infty\end{matrix}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{\frac{4}{x}}\cdot\frac{-4}{x^2}}{1} = \lim_{x\to\infty} \frac{-1}{x} = 0$$

26. *L'Hopital's for limits of type* 1^∞

Does the limit $\lim_{x\to\infty} \left(1+\frac{8}{x}\right)^x$ converge? If so, what does it converge to?

Option 1: recognize this limit! it is e^8.

Option 2:

$$\lim_{x\to\infty}\left(1+\frac{8}{x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{8}{x}\right)} = e^{\lim_{x\to\infty}\left(x\cdot\ln\left(1+\frac{8}{x}\right)\right)}$$

Sub-question

$$\lim_{x\to\infty} \underset{\infty}{x}\cdot\ln(\underset{0}{1+\tfrac{8}{x}}) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{8}{x}\right)}{\frac{1}{x}} \quad \begin{matrix}\nearrow 0 \\ \searrow 0\end{matrix}$$

$$\overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{1}{1+\frac{8}{x}}\cdot\left(\frac{-8}{x^2}\right)}{\frac{-1}{x^2}}\right) = \lim_{x\to\infty}\left(\frac{1}{1+\frac{8}{x}}\right)\cdot 8 = 8$$

$$= e^8$$