Review Sheet #2 | Math 141 | Series, and Power/Taylor Series

Name: Solutions Section: ______________________

1. *Test for Divergence*

Determine if the **series** $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n}{2}\right)$ converges or diverges.

Notice: $\lim_{n\to\infty} \tan^{-1}\left(\frac{n}{2}\right) = \frac{\pi}{2}$

$\Rightarrow$ the series diverges by divergence test

2. *Special Types of Series:* p-series

Determine if the **series** $\sum_{n=1}^{\infty} \frac{1}{n^{1.0000001}}$ converges or diverges.

converges: it is a p-series with $p > 1$

3. *Special Types of Series:* p-series

Determine if the **series** $\sum_{n=1}^{\infty} \frac{1}{n^{.9786}}$ converges or diverges.

diverges: it is a p-series with $p \leq 1$

4. *Special Types of Series:* geometric

Determine if the **series** $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{4n} \cdot 5^{1-2n}}{\pi^{n+2}}$ converges or diverges.

This is a geometric series: $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2^4)^n \cdot 5}{(5^2)^n \cdot \pi^n \cdot \pi^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^4 \cdot 5}{5^2 \cdot \pi \cdot \pi^2} \left(\frac{-2^4}{5^2\pi}\right)^{n-1}$

$|r| = \left|\frac{-2^4}{5^2\pi}\right| = \left|\frac{-4^2}{5^2\pi}\right| < 1 \Rightarrow$ the series converges.

5. *Special Types of Series:* alternating

Determine if the **series** $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+2}$ converges or diverges.

Notice: $\cos(n\pi)$ oscillates between 1 & -1

so this is an alternating series.

ALSO $\frac{1}{n+2}$ decreases to 0, so the series converges by alternating series test

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6. *Special Types of Series:* telescoping

Determine if the **series** $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converges or diverges.

Notice: the series telescopes: $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$

look at partial sums!

$$S_n = a_1 + \cdots + a_n = \begin{aligned} &\ln(1) - \cancel{\ln(2)} \\ &+ \cancel{\ln(2)} - \cancel{\ln(3)} \\ &+ \vdots \\ &+ \cancel{\ln(n-1)} - \cancel{\ln(n)} \\ &+ \cancel{\ln(n)} - \ln(n+1) \end{aligned} = \ln(1) - \ln(n+1)$$

return to the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\ln(1) - \ln(n+1)\right] = -\infty$$

$\Rightarrow$ the series diverges.

7. *Tests for Positive Series:* direct comparison test

Determine if the **series** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges.

compare to a smaller series to show divergence

notice: $\dfrac{\ln(n)}{n} \geq \dfrac{1}{n}$ & both are positive.

and $\sum \frac{1}{n}$ diverges: p-series w/ $p \leq 1$

So $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by direct comparison with $\sum \frac{1}{n}$

8. *Tests for Positive Series:* direct comparison test

Determine if the **series** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \cdot \ln(n)}$ converges or diverges.

compare to a larger series to show convergence

Notice: $\dfrac{1}{n^2 \cdot \ln(n)} \leq \dfrac{1}{n^2}$ (because $\ln(n) > 1$ for $n \geq 2$).

and both series are positive

and $\sum \frac{1}{n^2}$ converges (p-series, $p > 1$)

So $\sum \frac{1}{n^2 \cdot \ln(n)}$ converges by direct comparison with $\sum \frac{1}{n^2}$

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9. *Tests for Positive Series:* limit comparison test

Determine if the **series** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{\sqrt{n^5+1}}$ converges or diverges.

Notice $a_n = \dfrac{n^2+n+1}{\sqrt{n^5+1}} = \dfrac{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{n^{2.5}\sqrt{1+\frac{1}{n^5}}}$

a nonzero finite #

So if $b_n = \frac{1}{n^{.5}}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{\sqrt{1+\frac{1}{n^5}}} = 1$

by limit comparison test our series does the same thing as $\sum \frac{1}{n^{.5}}$. This is a p-series $p \le 1$ so diverges

$\Rightarrow$ Our series diverges by limit comparison to $\frac{n^2}{n^{2.5}}$

10. *Tests for Positive Series:* a direct comparison **followed by** a limit comparison

Determine if the **series** $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot \ln(n)}{(n+1)^4}$ converges or diverges.

Notice $\dfrac{n\cdot\ln(n)}{(n+1)^4} \le \dfrac{n^2}{(n+1)^4}$ (because $\ln(n) \le n$)

nonzero finite #

AND Notice: $a_n = \dfrac{n^2}{(n+1)^4} = \underbrace{\boxed{\dfrac{n^2}{n^4}}}_{=b_n} \cdot \dfrac{1}{(1+\frac{1}{n})^4}$, so $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^4} = 1$

because $\sum \frac{n^2}{n^4}$ converges (p-series $p>1$), then $\sum \frac{n^2}{(n+1)^4}$ converges

$\Rightarrow$ our series converges by direct comparison with $\sum \frac{n^2}{(n+1)^4}$

11. *Tests for Positive Series:* integral test

Determine if the **series** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln(n)}$ converges or diverges.

this is a positive series & a_n decreases to 0

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot\ln(n)}$ converges $\Leftrightarrow$ $\displaystyle\int_2^{\infty} \frac{1}{x\cdot\ln(x)}\,dx$ converges

But $\displaystyle\int_2^{\infty} \frac{1}{x\cdot\ln(x)}\,dx = \lim_{t\to\infty}\int_2^{t}\frac{1}{x\cdot\ln(x)}\,dx = \lim_{t\to\infty}\int_{\ln(2)}^{\ln(t)}\frac{1}{u}\,du$ diverges

So $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln(n)}$ diverges by the integral test.

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12. *The big guns:* ratio test

has factorial or n^n $\Rightarrow$ use ratio / root test

Determine if the **series** $\sum_{n=1}^{\infty} \frac{(2n+1)!}{(n!)^2}$ converges or diverges.

- $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1)+1)!}{(n+1)!\cdot(n+1)!}\cdot\frac{n!\cdot n!}{(2n+1)!}\right|$$

$$= \left|\frac{(2n+3)!}{(2n+1)!}\cdot\frac{n!}{(n+1)!}\cdot\frac{n!}{(n+1)!}\right|$$

$$= \left|\frac{(2n+3)(2n+2)\cancel{(2n+1)\cdots 2\cdot 1}}{\cancel{(2n+1)\cdots 2\cdot 1}}\cdot\frac{1}{(n+1)}\cdot\frac{1}{(n+1)}\right| = \frac{(2n+3)(2n+2)}{(n+1)(n+1)}$$

- $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{4}{1} > 0 \quad \Rightarrow$$ the series diverges by ratio test.

13. *The big guns:* root test

all terms have exponent n $\Rightarrow$ use root test

Determine if the **series** $\sum_{n=2}^{\infty} \frac{(2n+3)^n}{(n-1)^{2n}}$ converges or diverges.

$$= \sum_{n=2}^{\infty}\left(\frac{2n+3}{(n-1)^2}\right)^n$$

- $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{2n+3}{(n-1)^2}\right) = 0$$

$\Rightarrow$ the series converges (absolutely). by the root test

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14. *Sometimes negative series:* absolute convergence using the ratio test

Does the **series** $\sum_{n=1}^{\infty} (-1)^n \, n \, e^{-n}$ converge absolutely, converge conditionally, or diverge?

$$= \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n}$$

Can apply Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1}(n+1)}{e^{n+1}} \cdot \frac{e^n}{(-1)\,n} \right| = \frac{1}{e} \cdot \left(\frac{n+1}{n}\right)$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{e} < 1 \Rightarrow$ the series converges absolutely by the ratio test

15. *Sometimes negative series:* conditional convergence

Does the **series** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$ converge absolutely, converge conditionally, or diverge?

Notice: the series alternates sign AND $\frac{1}{\sqrt{2n+1}}$ decreases to 0

so $\boxed{\text{the series converges}}$ by alternating series test

But: $\sum \left| \frac{(-1)^n}{\sqrt{2n+1}} \right| = \sum \frac{1}{\sqrt{2n+1}}$ diverges, by limit comparison with $\frac{1}{\sqrt{n}}$

so $\boxed{\text{the series converges conditionally}}$ (try writing this step out!)

16. *Sometimes negative series:* absolute convergence using a test for positive series

Does the **series** $\sum_{n=1}^{\infty} \frac{n \cdot \sin(n)}{\sqrt{n^5+1}}$ (a_n) converge absolutely, converge conditionally, or diverge?

$\boxed{\text{we will show } \sum \left| \frac{n \cdot \sin(n)}{\sqrt{n^5+1}} \right| \text{ converges}}$

Notice: $0 \le \left| \frac{n \cdot \sin(n)}{\sqrt{n^5+1}} \right| \le \frac{n \cdot 1}{\sqrt{n^5+1}} = \frac{n}{n^{5/2}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^5}}}$

because $0 \le |\sin(n)| \le 1$

because $\lim_{n\to\infty} \frac{a_n}{\frac{n}{n^{5/2}}} = 1$, we can apply limit comparison: Because $\sum \frac{n}{n^{5/2}}$ converges (P-series $p>1$)

our series converges ABSOLUTELY

17. *Sometimes negative series:* divergence

Does the **series** $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ converge absolutely, converge conditionally, or diverge?

Notice: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[(-1)^n \frac{n^2-1}{n^2+1} \right] \neq 0$

Therefore the series diverges by the divergence test