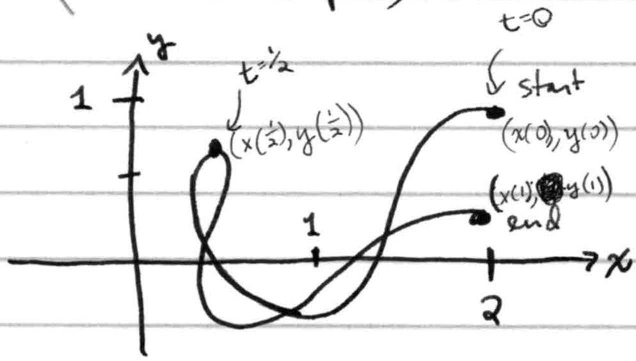
10.1 - Defining Fancy Curves by using equations with a parameter

Say:

Power Series give a nice way to represent our rich language of mathematical functions.
But our current language isn't (quite) rich enough for all interesting curves
For example, how can we describe this:

5 quick



Not a function: fails vertical line test

How can we describe this?

Say:

(Imagine you started a stopwatch when I started drawing)

Then We can make a table:

	t	$x(t)$	$y(t)$
start $t=0$ →	0	2	1
pretend: middle: $t=\frac{1}{2}$ →	$\frac{1}{2}$	~~2~~ $\frac{1}{2}$	$\frac{3}{4}$
pretend end $t=1$ min →	1	2	$\frac{1}{4}$

summary: we can represent curves by introducing a parameter t and keeping track of $x(t)$ and $y(t)$ separately

Eg: Let $x(t) = e^{t}$ for [redacted] between -1 and 1 [redacted]

$y(t) = e^{-t}$

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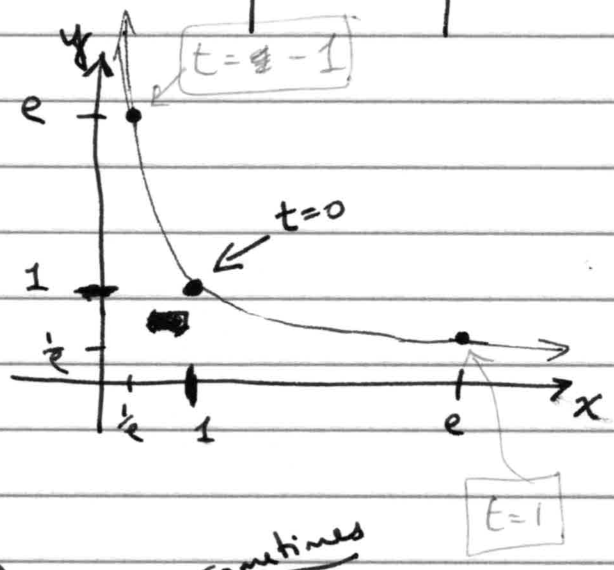
Ⅰ Graph [redacted] using a table

t	$x(t)$	$y(t)$
-1	$e^{-1} = \frac{1}{e}$	e
0	$e^{0} = 1$	$e^{-0} = 1$
1	$e^{1} = e$	$e^{-1} = \frac{1}{e}$



Ask: - which point drawn at $t=-1$?

$x(-1) = \frac{1}{e} < 1$

what point drawn at $t=1$?

$x(1) = e > 1$

Ⅱ We can sometimes find an equation by eliminating t

find x: $x(t) = e^{t}$

$y(t) = e^{-t} = \frac{1}{e^{t}} = \frac{1}{x(t)}$ $\Rightarrow$ $\boxed{y = \frac{1}{x}}$

here it is!

[redacted]

Some times

[redacted] we can eliminate t

Eg: $x(t) = e^{t} + 1$

$y(t) = e^{-t}$ [redacted] $+ e^{t}$

$\Rightarrow \quad x = e^{t} + 1 \qquad\qquad y = \dfrac{1}{e^{t}} + e^{t}$

[redacted]

7 $e^{t} = x - 1 \qquad\qquad \boxed{y = \dfrac{1}{(x-1)} + (x-1)}$

[redacted]

No t.

Other times

[redacted] we can't eliminate t

Eg: $x(t) = e^{-t} +$ [redacted] t

$y(t) = e^{t} - t$

[See the online homework.

you will see why when you see the graph]

Eg: Graph $x = \cos(t)$
$y = \sin(t)$

for $0 \le t \le 2\pi$

Compute:

t	$x(t)$	$y(t)$
0	1	0
$\pi/2$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

NOTICE: the same point is graphed by TWO DIFFERENT t's

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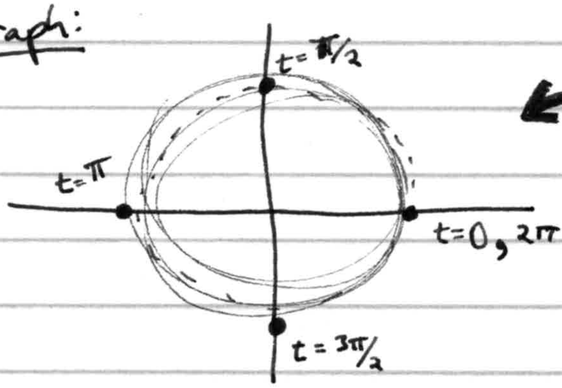
Graph:



Say: think: this looks like a circle!

or

use/remember: trigonometry to show it is a circle.

Can we eliminate t?

can't just "find x".

use what we know!

know $\underbrace{(\sin(t))}_{\text{"}y\text{"}}^2 + \underbrace{(\cos(t))}_{\text{"}x\text{"}}^2 = 1$

So: $y^2 + x^2 = 1$ ← we can eliminate t here.

To graph More Complex Equations

could do this in either order

① graph $x(t)$ vs t
   and $y(t)$ vs t

② make a table of nice values

③ use $\overset{\uparrow x}{\longrightarrow t}$ and $\overset{\uparrow y}{\longrightarrow t}$ to fill in the gaps

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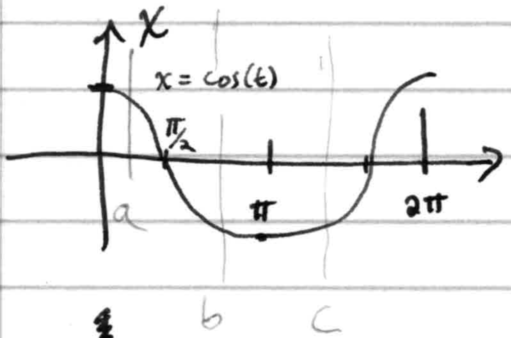
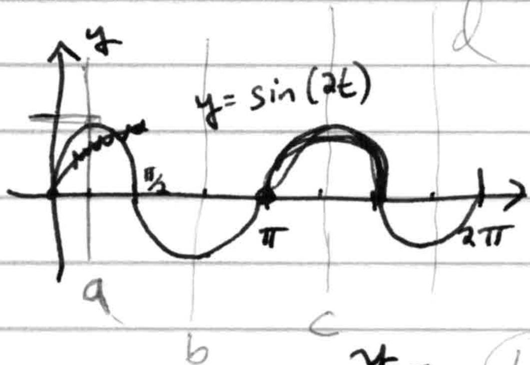
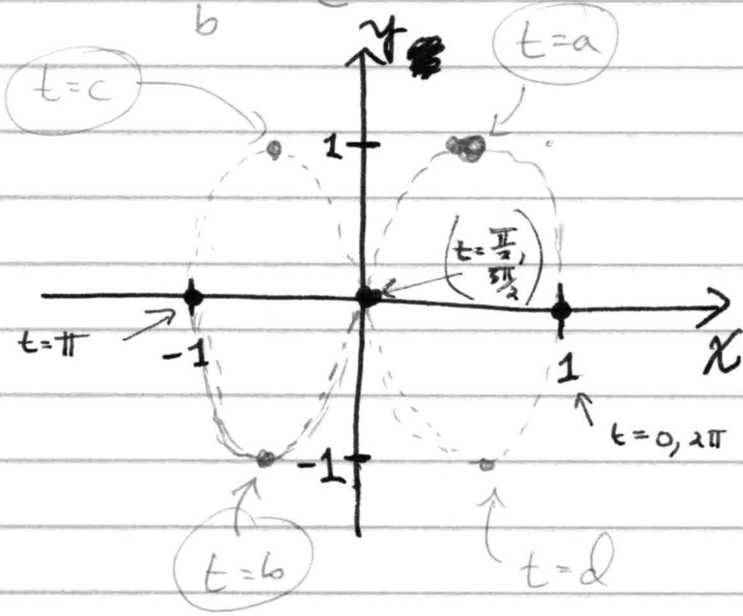
Eg: graph $x = \cos(t)$ for t in $[0, 2\pi]$
$y = \sin(2t)$





t	$x(t)$	$y(t)$
0	1	0
$\frac{\pi}{2}$	0	0
π	-1	0
$\frac{3\pi}{2}$	0	0
2π	1	0
a	positive # less than 1	1
b	neg # greater than −1	−1
c	neg # > −1	1
d		



Calculus with Parametric Curves:

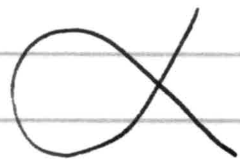
Yesterday, we looked at a new way of defining a curve: we introduced a parameter t and kept track of $x(t)$ & $y(t)$ separately

Today, we look at the slopes of parametric curves.

Hopefully: this can help us graph curves slightly more accurately.

(7)

Consider the Curve



Notice: [y changes as x changes

even though y isn't a function of x...

y looks like a function If you zoom in enough

y changes as t changes

x changes as t changes

[so (1) we can think of $y(t)$ as a function of x
and (2) x is a function of t]

By the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

When $\frac{dx}{dt} = 0$, one of 2 strange things happen

- If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

  The curve has a vertical tangent line
  (the pen drawing the curve is NOT moving side to side, and IS moving up/down)

- If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

  The pen drawing the curve has (briefly) stopped moving

(10)

When $\frac{dx}{dt} = 0$, (we can solve the earlier formula for $\frac{dy}{dx}$)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

remember!

- try thinking "dt's cancel"
- DON'T FORGET $\frac{dy}{dt}$ goes on TOP

Eg: Consider the curve $x(t) = t^3 + 2$

$y(t) = t^2 + t + 1$

Find $\frac{dy}{dt}$ at the point $(3, 3)$

① find all t that plots this point.

$$x(t) = t^3 + 2 = 3$$

$$t^3 = 1$$

$$\Rightarrow t = 1$$

(10)

Check

$y(1) = 1^2 + 1 + 1 = 3$ ✓

② find $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ ← y is on top of the fraction.

$$\frac{dy}{dt} = 2t + 1$$

$$\frac{dx}{dt} = 3t^2$$

$\boxed{\text{NOTICE } \frac{dx}{dt} \neq 0 \text{ for } t = 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{2t+1}{3t^2} \begin{matrix} \leftarrow \frac{dy}{dt} \\ \leftarrow \frac{dx}{dt} \end{matrix}$$

③ compute $\boxed{\frac{dy}{dt} = \frac{2(1)+1}{3(1)^2} = \frac{3}{3} = 1}$

when $t=1$ and/or $\boxed{\text{at } (3,3)}$

SKIP IN CLASS

There are >3 pgs on this eqn

Eg: let C be $x(t) = t^2$ for all t

$y(t) = t^3 - 3$

Find the points (x,y) on C with horizontal and vertical tangents

we want to use the $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ formula

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2 - 3$$

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moving sideways — NOT moving vertically

Horizontal when: $\boxed{\frac{dx}{dt} \neq 0}$ and $\boxed{\frac{dy}{dt} = 0}$

(skip)

$\frac{dy}{dt} = 0$ when $3t^2 - 3 = 0$

$3t^2 = 3$

$t = \pm 1$

to find (x,y), plug in t

$t = 1 \Rightarrow x(1) = 1$

$y(1) = 1^3 - 3 \cdot 1 = -2$

$\Rightarrow$ point is $(1, -2)$

$t = -1 \Rightarrow x(-1) = 1$

$y(-1) = \cdots = -2$

$\Rightarrow$ the point is $(1, 2)$

---

moving up/down — not moving sideways

Vertical when: $\boxed{\frac{dy}{dt} \neq 0}$ and $\boxed{\frac{dx}{dt} = 0}$

$\frac{dx}{dt} = 0$ when $t^2 = 0$, so $t = 0$, $\Rightarrow$ the point is $(0,0)$

SKIP IN CLASS | If <20 min

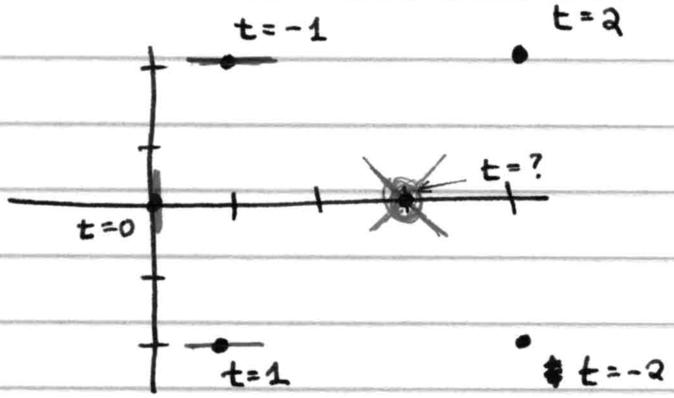
To Graph $\quad x = t^2$

$\qquad\qquad\quad y = t^3 - 3t$

make a table

t	$x(t)$	$y(t)$
-2	4	$(-8)-3(-2) = -2$
-1	1	$(-1)-3(-1) = 2$
0	0	0
1	1	$1^3 - 3 = -2$
2	4	$8 - 6 = 2$

(10)



to fill in details, find slopes at these (or other) points

horizontal at $(1,-2)$ and $(1,2)$

vertical at $(0,0)$

Hint: the graph intersects twice at $(3,0)$

To finish the picture

Find the two tangent lines at $(3,0)$

① for what t is $\big(x(t), y(t)\big) = (3,0)$?

$$x(t) = t^2 = 3 \iff t = \pm\sqrt{3}$$

check:

$$y(\pm\sqrt{3}) = (\pm\sqrt{3})^3 - 3\cdot(\pm\sqrt{3}) \qquad 3 = (\sqrt{3})^2$$

$$= \pm(3)^{\frac{3}{2}} - (3^{3/2}) = 0$$

(10) ② find slope at $t = \sqrt{3}$

and $t = -\sqrt{3}$

use

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^3 - 3t) = 3t^2 - 3$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2) = 2t$$

at $t = \sqrt{3}$ $\quad \frac{dy}{dx} = \frac{3(\sqrt{3})^2 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \boxed{\sqrt{3}}$

at $t = -\sqrt{3}$ $\quad \frac{dy}{dx} = \frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = \frac{6}{-2\sqrt{3}} = \boxed{-\sqrt{3}}$

③ find the tangent lines

$t =$ [redacted] $\sqrt{3}$

x-coordinate   y-coordinate

$$y = \sqrt{3}(x - 3) - 0$$

$$= \sqrt{3}x - 3\sqrt{3}$$

$t = -\sqrt{3}$

(3) $y = -\sqrt{3}(x-3) - 0$ [redacted] [redacted]

$$= -\sqrt{3} \cdot x + 3\sqrt{3}$$

Calculus with Parametric Curves II

AND, Polar I

last time, we computed the slopes of parametric curves
and took a second shot at graphing them

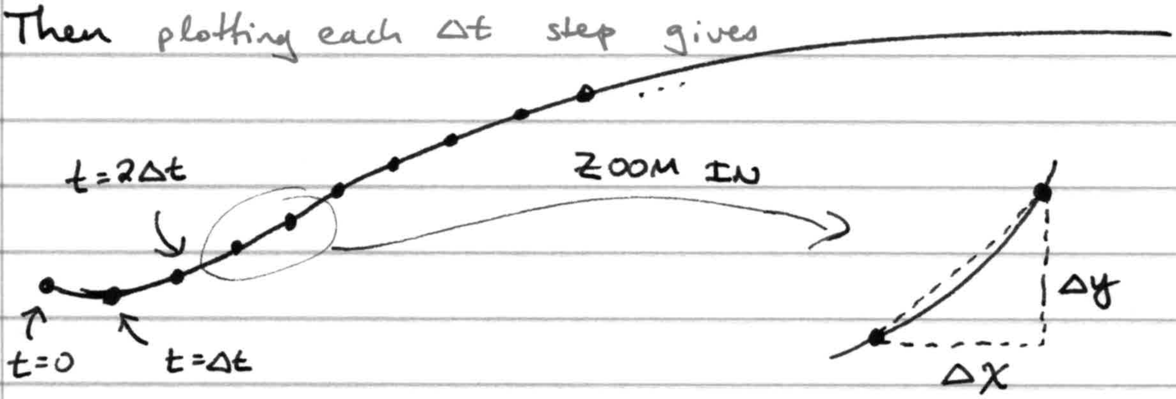
this time: we'll compute the length of a parametric curve

Given a drawn curve: its very easy to find the length:
put a piece of string on it, cut to length,
and measure with a ruler!

Ok, that's not a great option,
(5) but it gives the right idea:
approximate the curve with something easier to measure

Idea: chop the "t-axis"
into small lengths Δt

Then plotting each Δt step gives



length of piece $= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Intuitively:

$$\Delta x = \frac{dx}{dt} \cdot \Delta t$$

The textbook gives a more precise argument

$$\Delta y = \frac{dy}{dt} \cdot \Delta t$$

$$\text{length of curve} = \overbrace{\lim_{\Delta t \to 0}}^{\text{for smaller \& smaller time steps}} \overbrace{\sum \sqrt{(\Delta x)^2 + (\Delta y)}}^{\leftarrow \cdots\ \text{sum of all the pieces}}$$

(10)

$$= \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{dx}{dt} \cdot \Delta t\right)^2 + \left(\frac{dy}{dt} \cdot \Delta t\right)^2}$$

(pull out of the sqrt)

$$= \underbrace{\lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t}_{\text{this is a Reimann sum!}}$$

$$\boxed{\text{arc's length} = \int_{\text{starting } t}^{\text{ending } t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$$

↑ a nice formula!

Aside: another shape would be nice (eg ellipse)
BUT can't evaluate integral for most shapes (eg. ellipse) ← not even wolfram

Eg: Find the Arc-length of the radius 2 circle

$$x(t) = 2\cdot \cos(t)$$
$$y(t) = 2\cdot \sin(t)$$

for $\boxed{0 \le t \le 2\pi}$

starting & ending t does matter for arc lengths.

$$\text{Arc-length} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(10)

$$= \int_0^{2\pi} \sqrt{(-2\cdot \sin(t))^2 + (2\cdot\cos(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\cdot \sin^2 t + 4\cdot \cos^2 t}\, dt$$

$$= \int_0^{2\pi} (\sqrt{4}) \cdot \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} 2\, dt$$

$$\text{Arc-length} = 4\pi$$

Start & End t DOES matter for arc-length

Because the curve for $0 \le t \le 2\pi$ draws exactly the circle, its circumference $= 4\pi$

NOTE the same curve for $0 \le t \le \boxed{4\pi}$ traces the circle TWICE, & has arc-length 8π

10.3 - Polar Coordinates

parametric equations keep track of x & y coordinates separately.

This is very expressive, but is often very cumbersome to work with

Today, we'll make the shift from cartesian coordinates to polar coordinates

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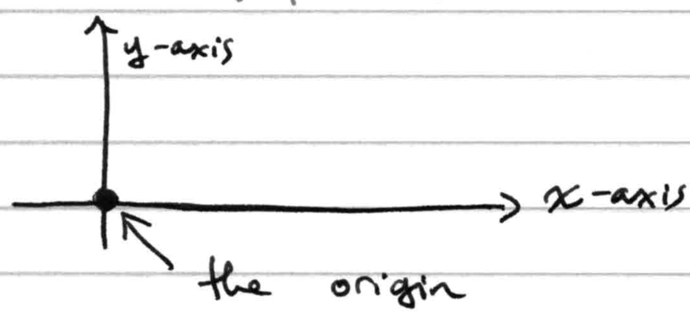
Formally: Cartesian coordinates start with two perpindicular lines and a point

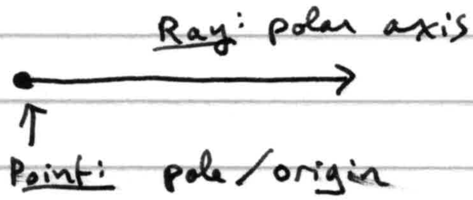
5 [redacted] ONLY



and you keep track of "distance to the side" and "distance up"

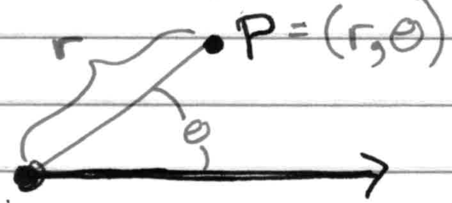
Often, it is more natural to work with "direction & distance"

Polar Coordinates start with one line and one point



To plot a point, specify

1. distance from the origin/pole
2. ∡ from the polar axis

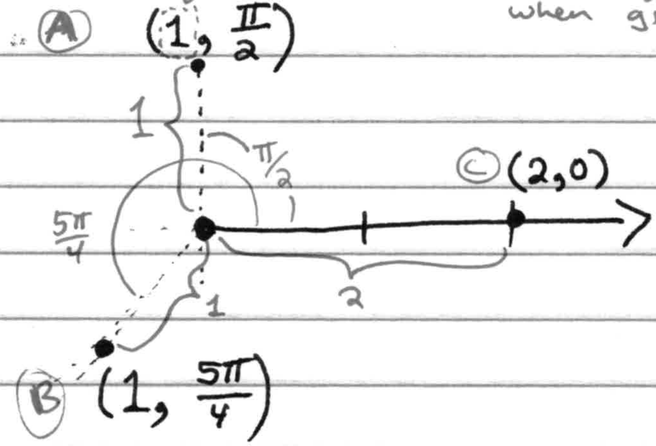


5 ONLY

Eg: plot $(1, \frac{\pi}{2})$, $(1, \frac{5\pi}{4})$, $(2, 0)$

A B C

r comes first, even though you think about θ first when graphing



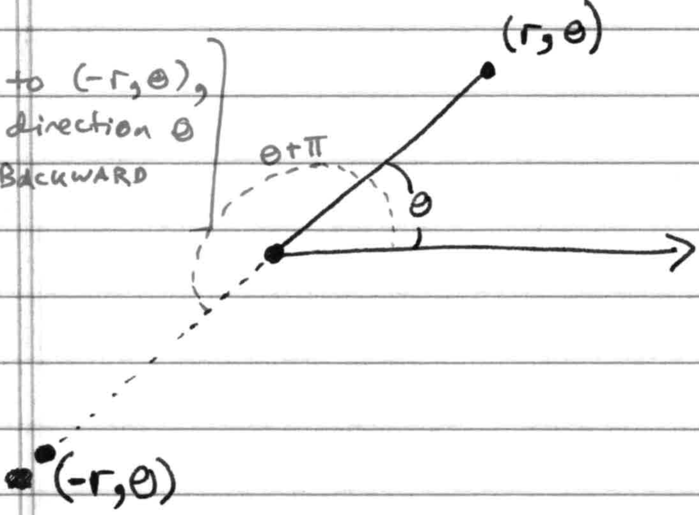
Notice: each point has <u>many</u> polar representations

$(2,0) = (2, 2\pi) = (2, -4\pi)$

We also allow Negative #'s as lengths

If r is **positive** get to r, θ by pointing in direction θ & going FORWARD r-unit

[To get to $(-r, \theta)$, point in direction θ and go BACKWARD r-unit]



[redacted]

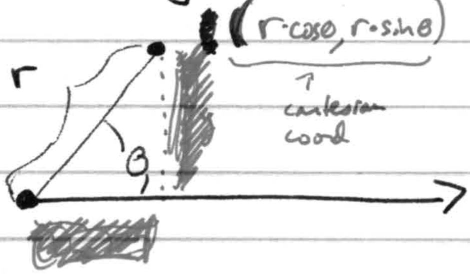
Notice:

$(-r, \theta)$ and $(r, \theta+\pi)$ are the same point.

$\boxed{\leq 5 \text{ ONLY}}$

(You will have plenty of chances to play with this on the homework)

Converting from polar to (x,y) is easy



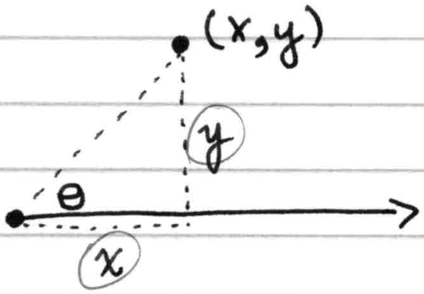
[redacted] What is x- and y- coordinate?

$$x = r \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta)$$

(10) **Eg:** $\underbrace{\left(1, \frac{\pi}{4}\right)}_{\text{in polar}}$ is $\underbrace{\left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\text{in cartesian}}$

Converting from (x,y) to polar takes thought



Need: distance r & θ

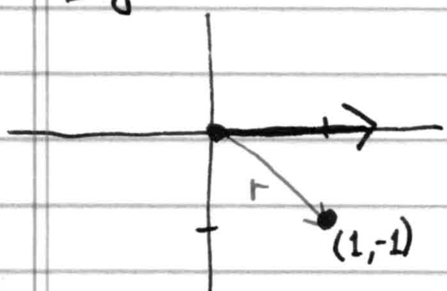
looking at $\triangle$,

$$r = \sqrt{x^2 + y^2}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

But YOU must choose θ in the correct quadrant

Eg: write cartesian $(1,-1)$ as polar



$$\boxed{r = \sqrt{1^2 + (-1)^2} = \sqrt{2}}$$

$$\tan\theta = \frac{y}{x} = -1$$

(happens for $\theta = \frac{3\pi}{4}$ AND $\theta = \frac{7\pi}{4}$)

$(-1,1)$ is in quadrant IV [redacted]

so $\theta = \frac{7\pi}{4}$ (or $\boxed{\theta = \frac{-\pi}{4}}$)

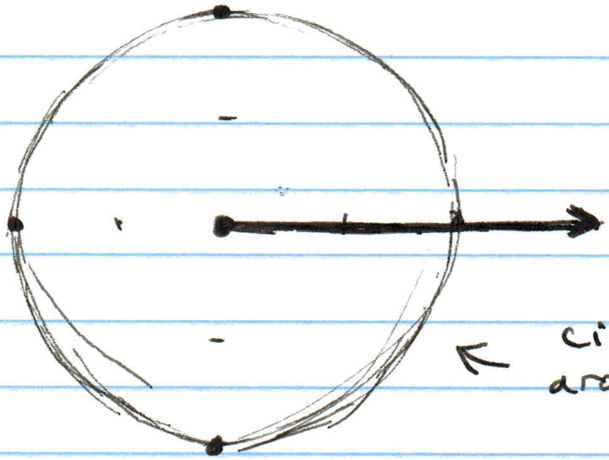
Graphing Polar Equations

Key Idea:
- let θ sweep out $\measuredangle$'s
  (usually between 0 & 2π is enough)
- For each θ, what is r?

Quick to graph:

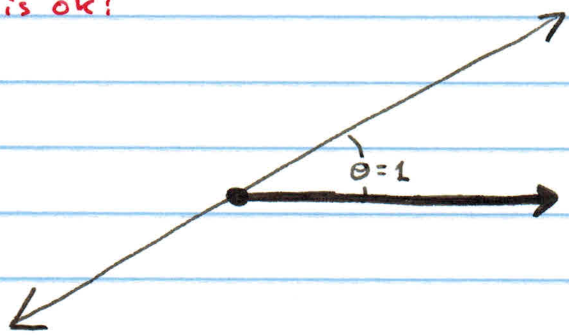
for all θ, $r = 2$

(5)



circle of radius 2 around the origin

$\theta = 1$ and any r is ok!



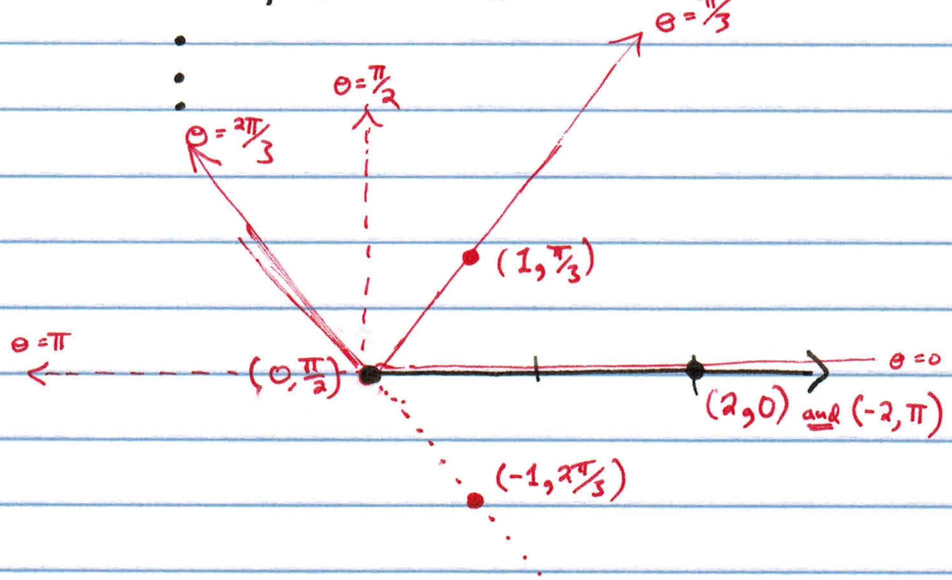
Eg: Let $r = 2 \cdot \cos\theta$ (for θ between 0 & π)

Ⓐ Graph points using a table

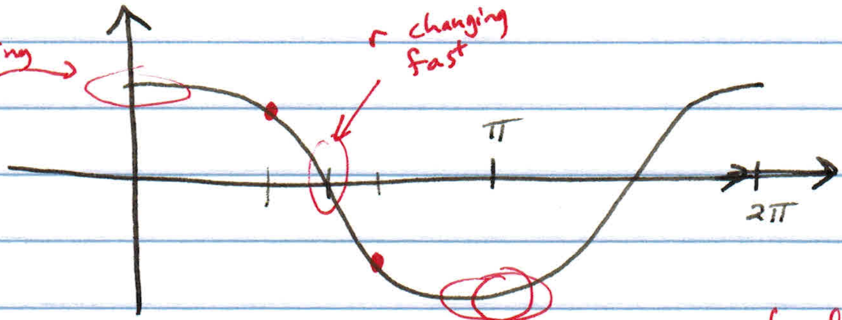
(leave space & pick 2nd)

θ	r
0	$2 \cdot 1 = 2$
$\pi/3$	$2 \cdot \frac{1}{2} = 1$
$\pi/2$	$2 \cdot 0 = 0$
$2\pi/3$	$2 \cdot \frac{-1}{2} = -1$
π	$2 \cdot -1 = -2$
⋮	

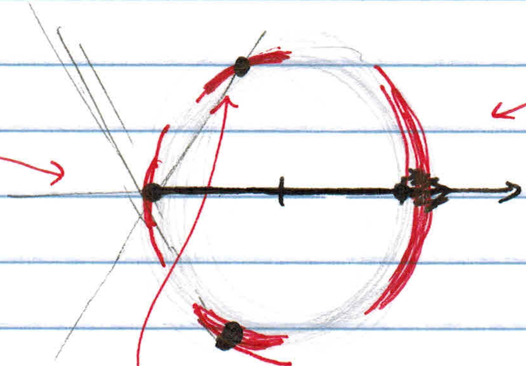
(5)-(10)



Ⓑ fill in the blanks using



[redacted] (5)-(10)



as θ goes from $<\frac{\pi}{2}$ to $>\frac{\pi}{2}$, r rapidly drops then rapidly becomes big neg #

as θ is near 0 & π r doesn't change much, so this side looks like a fixed radius circle

as θ nears $\frac{\pi}{3}$ [redacted] r changes (decreases) with increasing speed.

(The analogous thing happens for θ near $\frac{2\pi}{3}$)

this looks like the radius 1 circle around polar (1, 0) [redacted]

(10)

Ⓒ Graph by obtaining Cartesian Equation

If $r = 2 \cdot \cos(\theta)$

Then $x(\theta) = r \cdot \cos\theta = (2 \cdot \cos\theta) \cdot \cos\theta$
$= 2 \cdot \cos^2\theta$

$y(\theta) = r \cdot \sin\theta = (2 \cdot \cos\theta) \cdot \sin\theta$
$= 2 \cdot \sin\theta \cdot \cos\theta$

using trig identities

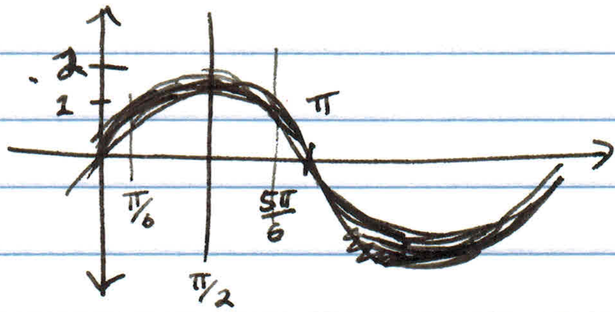
$$\left.\begin{array}{l} x = \cos(2\theta) + 1 \\ y = \sin(2\theta) \end{array}\right\} \text{this is a parametric equation!}$$

using pythagorean identity

$$y^2 + (x-1)^2 = 1$$

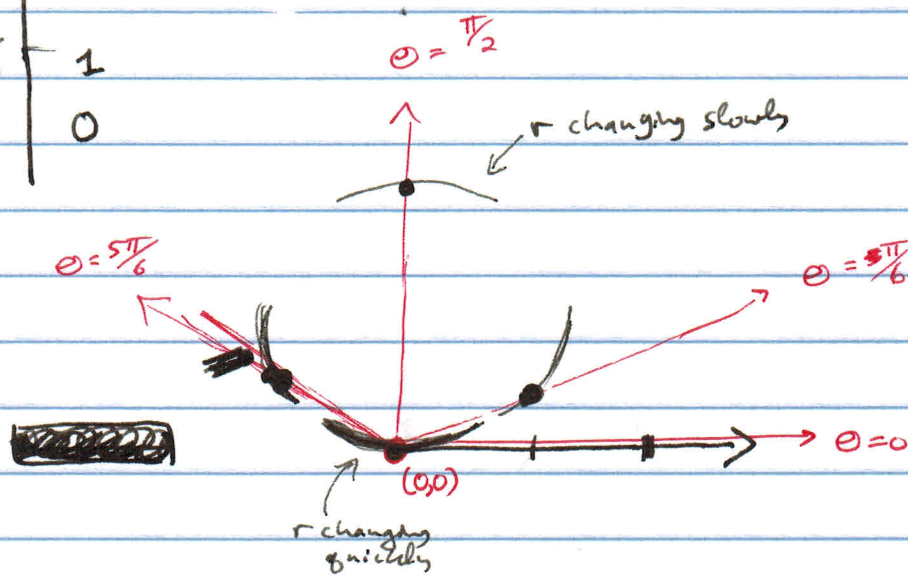
$\Rightarrow$ our curve really is a circle of radius 1 centered at $(1, 0)$

Eg: $r = 2\cdot\sin\theta$ is also a circle



θ	r
0	0
$\pi/6$	1
$\pi/2$	2
$5\pi/6$	1
π	0

(5)



and fill in gaps as before...

This circle is centered at

polar $\left(1, \frac{\pi}{2}\right)$

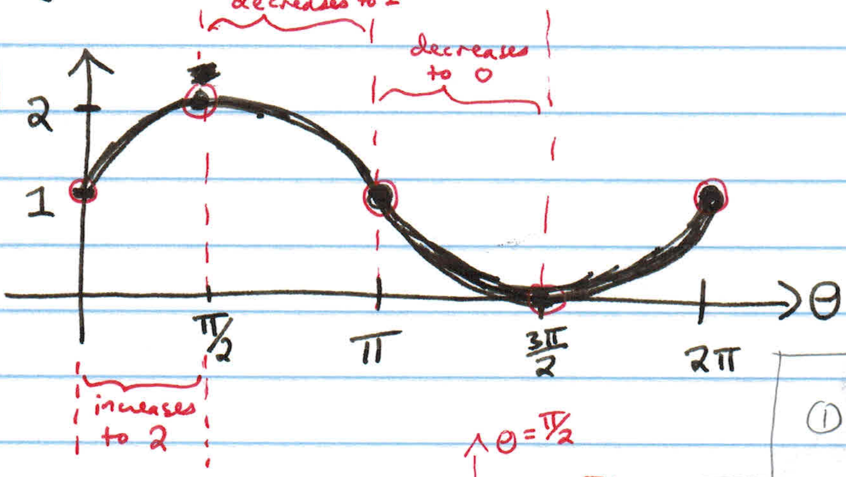
cartesian $(0, 1)$

$r = 1 + \sin(\theta)$

Eg: [redacted] is a Cardioid

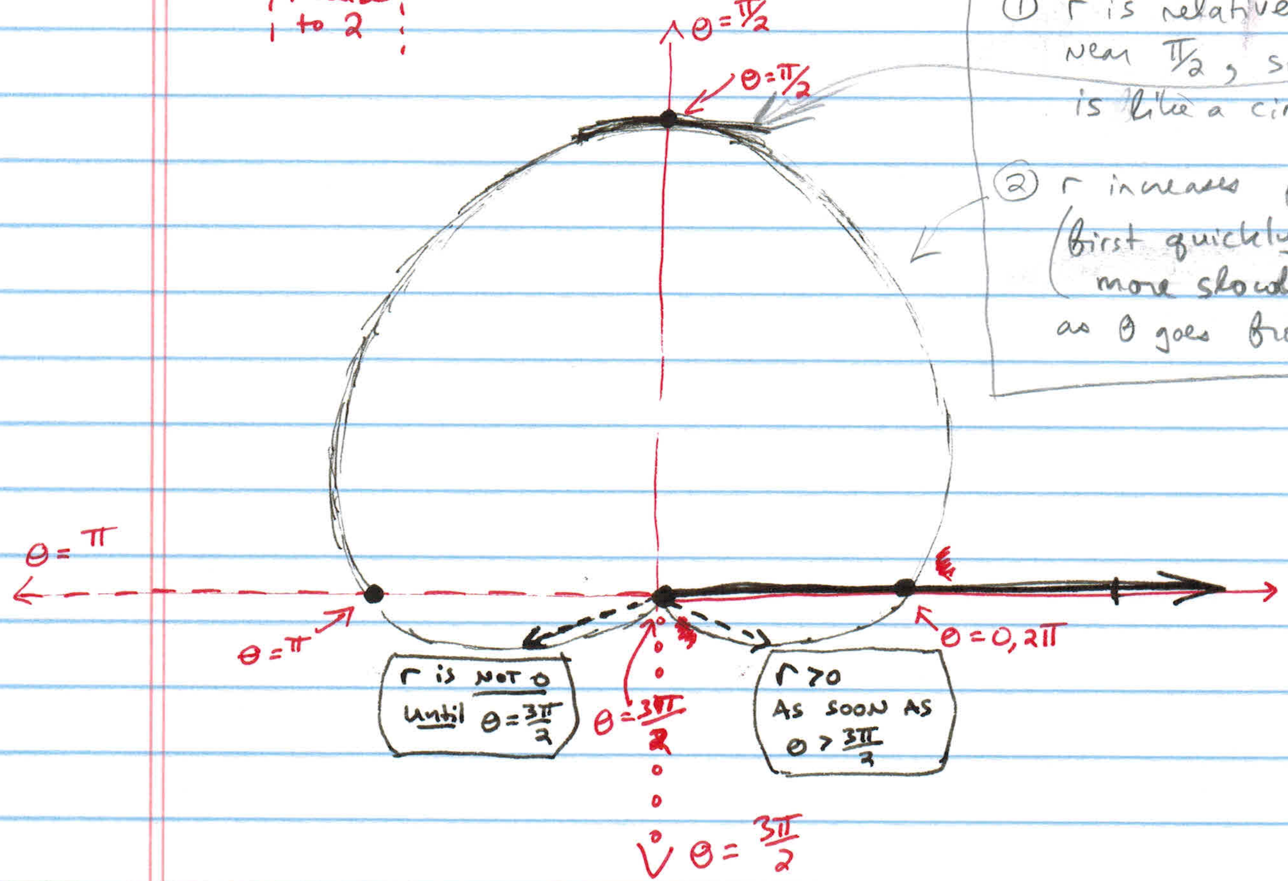
the graph of r is the graph of sin shifted up by 1

graph the 4 dots first to calibrate

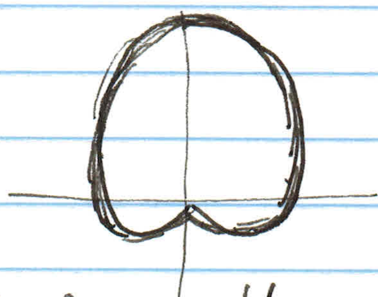


① r is relatively constant near $\frac{\pi}{2}$, so this is like a circle around (

② r increases from 1 to 2 (first quickly, then more slowly) as θ goes from 0 to $\frac{\pi}{2}$

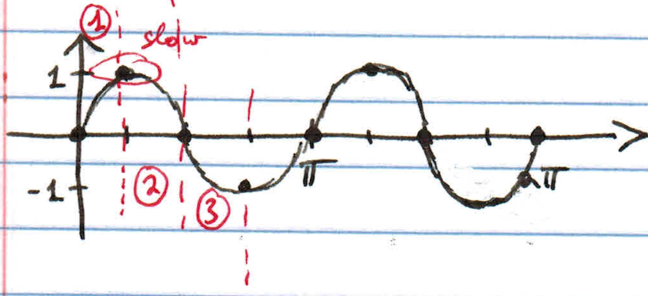


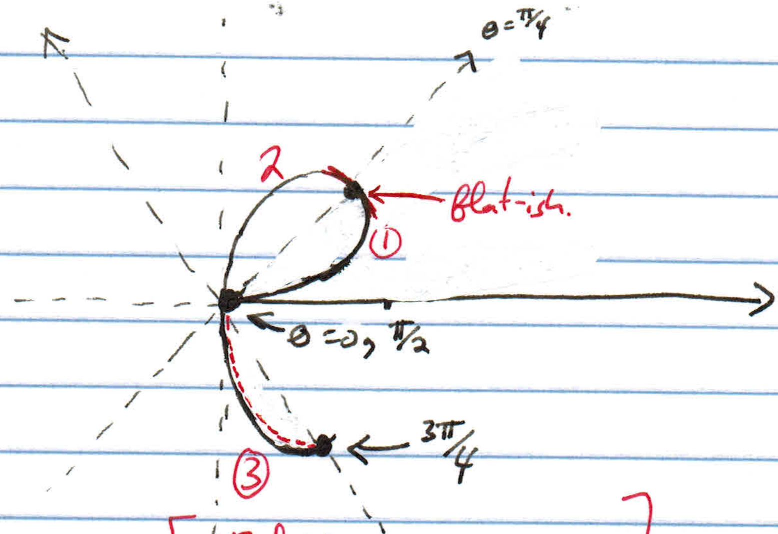
a nicer picture:



this is an upside-down heart!

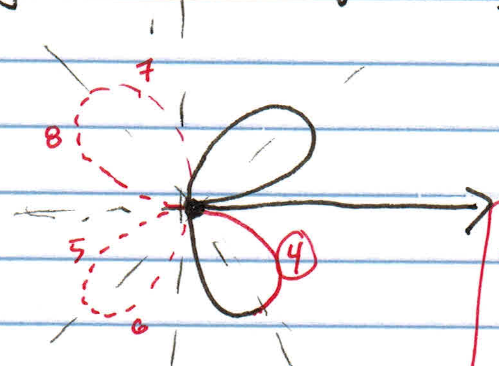
4-leaf Rose: $r = \sin(2\theta)$





- r decreases from 0 to -1 on $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$
- starts out changing fast then slows down

Starting new picture for clarity



draw 4 carefully (up to π)

then fill in rest (dashed) quickly

First: a cute idea
for going between polar & x-y equations.

Find a polar formula for

$$x^2 - y^2 = 1 \quad \leftarrow \text{a hyperbola}$$

(10)

remember: $x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$

$$(r \cdot \cos\theta)^2 - (r \cdot \sin\theta)^2 = 1$$

$$r^2 (\cos^2\theta - \sin^2\theta) = 1$$

$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$$

Remember double ∠ formulas!

$$\boxed{r^2 = \frac{1}{\cos(2\theta)} = \sec(2\theta)}$$

Same idea works backwards:

$$r^2 (\sin(2\theta) + \cos^2\theta) = 1$$

$$r^2 \cdot 2 \cdot \sin\theta \cdot \cos\theta + r^2 \cos^2\theta = 1$$

Pair up r's w/ sin & cos!

$$2(r \cdot \sin\theta)(r \cdot \cos\theta) + (r \cdot \cos\theta)^2 = 1$$

$$\boxed{2 \cdot y \cdot x + x^2 = 1} \leftarrow \text{(cartesian equation)}$$

A cartesian Equation for the 4-leaf Rose

Remember

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

$$r^2 = x^2 + y^2$$

so:

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$r = \pm\sqrt{x^2+y^2}$$

This helps us find an equation for our 4-leaf rose

$$r = \sin(2\theta) = 2 \cdot \sin\theta \cdot \cos\theta$$

$$r = 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$r^3 = 2 \cdot y \cdot x$$

$$\left(\pm\sqrt{x^2+y^2}\right)^3 = 2 \cdot y \cdot x$$

$$\boxed{\pm\left(x^2+y^2\right)^{\frac{3}{2}} = 2 \cdot y \cdot x}$$

the cartesian equation for the 4-leaf rose

Calculus and Polar Equations

[redacted]

Tangents to Polar Curves

If $r = f(\theta)$,

how do you find $\frac{dy}{dx}$

if there are no y's & x's ?

(5-10) Silly answer: graph curve, then use a protractor & plumb line.

Serious answer:

Notice: You can think of a polar curve as a parametric curve (with parameter θ)

Idea:

$$\boxed{\begin{aligned} x &= r \cdot \cos\theta = f(\theta)\cdot \cos(\theta) \\ y &= r \cdot \sin\theta = f(\theta)\cdot \sin(\theta) \end{aligned}}$$

so

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \text{as long as } \frac{dx}{d\theta} \neq 0}$$

Eg: for the Cardioid

$$r = 1 + \sin(\theta)$$

$$x(\theta) = \Big(\underbrace{1 + \sin(\theta)}_{r}\Big) \cdot \underbrace{\cos(\theta)}_{\cos\theta}$$

$$= \cos\theta + \sin\theta \cdot \cos\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -\sin\theta + \big(\cos\theta \cdot \cos\theta + \sin\theta(-\sin\theta)\big)$$

(10) and

$$y(\theta) = \Big(\underbrace{1 + \sin\theta}_{r}\Big) \cdot \underbrace{\sin\theta}_{\sin\theta}$$

$$= \sin(\theta) + \sin^2\theta$$

$$\Rightarrow \frac{dy}{d\theta} = \cos\theta + 2 \cdot \sin\theta \cdot \cos\theta$$

$\boxed{\text{So}}$ $$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

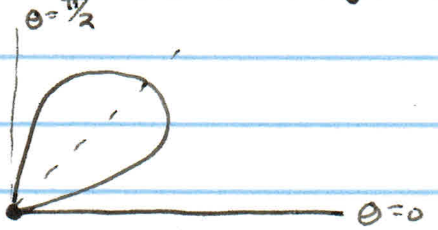
$$\frac{dy}{dx} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos^2\theta - \sin^2\theta} = \frac{\cos\theta + \sin(2\theta)}{-\sin(\theta) + \cos(2\theta)}$$

when $\frac{dx}{d\theta} \neq 0$

10.4 - Area of Polar Curves

Remember: $r = \sin(2\theta)$ was a 4-leaf rose

What is the area of one leaf?



(5)

Silly method:
- cut out 1 leaf,
- weigh it,
- and multiply by (paper density) · (paper thickness)

Serious method:

If $r = f(\theta)$ is a polar curve,

$$\text{Area of region swept out} = \int_{\text{starting } \theta}^{\text{ending } \theta} \frac{1}{2} r^2 \, d\theta = \int_{\text{starting } \theta}^{\text{ending } \theta} \frac{1}{2} [f(\theta)]^2 \, d\theta$$

[we'll look at an explanation of the formula if we have time on Monday]

For now, lets do a computation:

Eg: for the rose $r = \sin(2\theta)$

$$\text{Area of shaded leaf} = \int_{0}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_{0}^{\frac{\pi}{2}} \frac{1}{2}\left(\sin(2\theta)\right)^2 d\theta$$

$\pi/2$ ← θ where the leaf ends

0 ← θ where the leaf starts

(10)

$$= \int_{0}^{\pi/2} \frac{1}{2} \sin^2(2\theta) \, d\theta$$

[Now shift gears:
This is just another trig integral!]

$$\left(\sin^2(2\theta) = \frac{1 - \cos(2 \cdot 2\theta)}{2} \right)$$

$$= \int_{0}^{\pi/2} \frac{1}{4}\left(1 - \cos(4\theta)\right) d\theta$$

$$= \cdots = \left[\frac{\theta}{4} - \frac{\sin(4\theta)}{16} \right]_{0}^{\pi/2}$$

$$= \cdots = \boxed{\frac{\pi}{8} = \text{Area of one leaf}}$$

(10)

To find the area inside one curve and outside another

(1) sketch the curves & shade the region

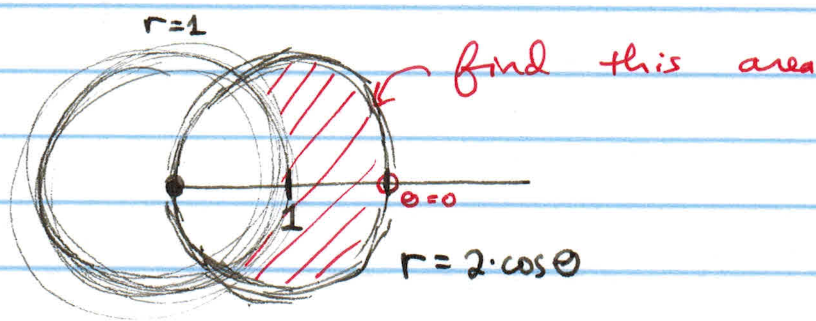
(2) Determine: curves intersect when $\underline{\theta = \text{start}}$ and $\underline{\theta = \text{end}}$

(3) set up formula and integrate

Eg: Find the area inside $r = 2\cdot\cos\theta$ and outside $r = 1$

① Remember/check these are both circles.

(r = 1 is easy) (to find orientation of $r = 2\cos(\theta)$ remember $r = 2\cdot\cos(0) = 2$ when $\theta = 0$)



② curves intersect when

$$1 = 2\cdot\cos\theta \iff \frac{1}{2} = \cos\theta \iff \theta = \frac{-\pi}{3} \text{ and } \theta = \frac{\pi}{3}$$

③ shaded area = $\boxed{\text{area of outside} - \text{area of inside}}$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cdot\cos\theta)^2\,d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(1)^2\,d\theta = \ldots$$

Calculus with Polar Curves 2

Remember: if $r = f(\theta)$ is a polar curve,

(5) $$\boxed{\text{Area of region swept out} = \int_{\text{starting }\theta}^{\text{ending }\theta} \frac{1}{2} r^2 \, d\theta = \int_{\text{starting }\theta}^{\text{ending }\theta} \frac{1}{2} [f(\theta)]^2 \, d\theta}$$

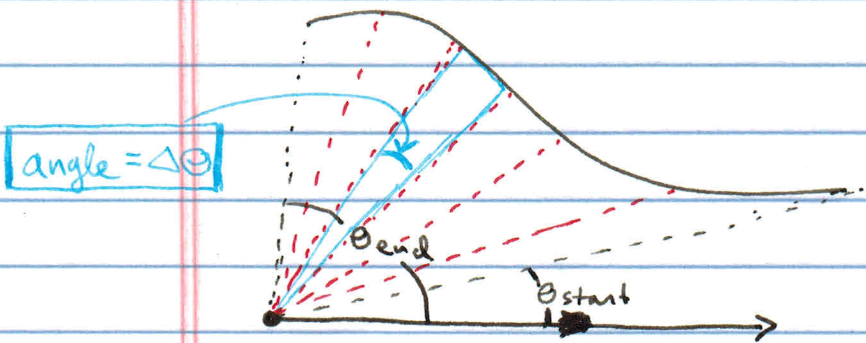
This is a nice formula, but

Where does this come from?

The proof uses 3 general ideas:

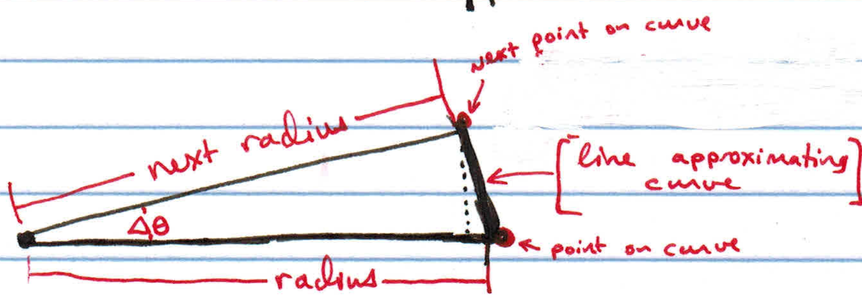
Where does the Polar Area formula come from?

Idea 1: to approximate area, cut θ-axis into small slices



(10)

Idea 2: Zoom in and approximate slice w/ Δ



$$\text{Area of } \Delta = \tfrac{1}{2} \cdot b \cdot h \text{ [redacted]}$$
$$= \tfrac{1}{2}(\text{radius})(\text{next-radius} \cdot \sin(\Delta\theta))$$

for small $\Delta\theta$, radius $\approx$ next-radius

$\Delta\theta \approx \sin(\Delta\theta)$

so: for small $\Delta\theta$, Area of $\Delta \approx \frac{1}{2}(\text{radius})^2 \cdot \Delta\theta$

Idea III: Area $\approx \sum$ area of slices

$$\approx \sum \frac{1}{2} r^2 \Delta\theta$$

In fact, can prove

(5) $$\text{Area} = \lim_{\Delta\theta \to 0} \sum \frac{1}{2} r^2 \Delta\theta$$

so $$\boxed{\text{Area} = \int_{\text{starting } \theta}^{\text{ending } \theta} \frac{1}{2} r^2 \, d\theta}$$

What is the take-away message?

$\frac{1}{2} r^2 \Delta\theta$ is the area of the small $\triangle$ slice

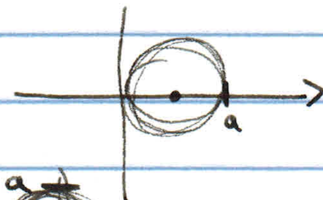
To compute "area outside $\boxed{\quad}$ but inside $\boxed{\quad}$" it is helpful to be familiar with common curve shapes

Cute Polar Graphs to know:

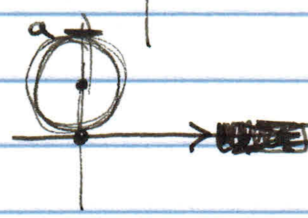
line: θ = some fixed $\measuredangle$

circles: $r = a$

$r = a \cdot \cos\theta$



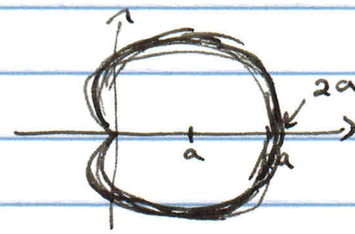
$r = a \cdot \sin\theta$



to remember which is which, plot r for $\theta = 0, \frac{\pi}{2}$

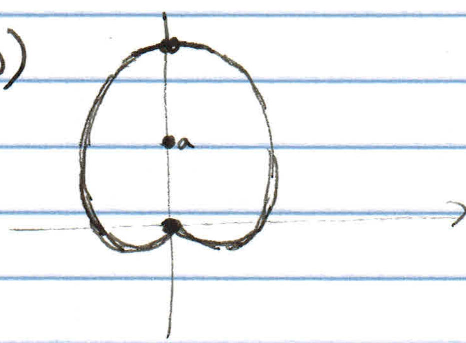
Cardioid:

$r = a(1 \pm \cos\theta)$



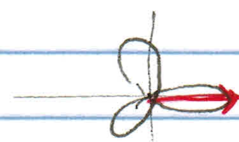
or

$r = a(1 \pm \sin\theta)$



to figure out orientation plot $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

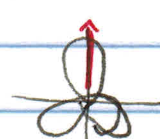
Roses: $r = a \cdot \cos(n\theta)$



$r = \cos(3\theta)$

cos always symmetric about horizontal

$r = \sin(3\theta)$



$r = a \cdot \sin(n\theta)$

to figure which is which, plot $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

sin always symmetric about vertical

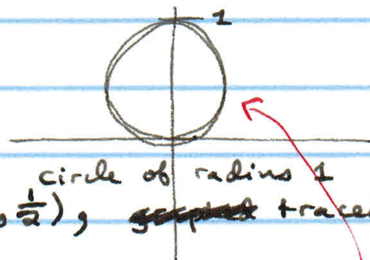
The textbook explores $r = 1 + c \cdot \sin(\theta)$ for different #'s c.

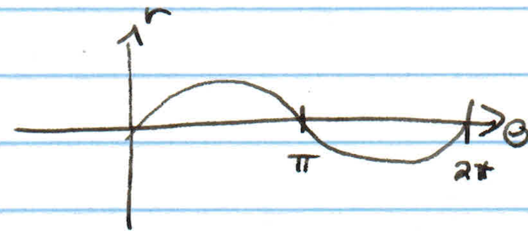
Let's Explore

$$r = a + \sin\theta \quad \text{on} \quad 0 \le \theta \le 2\pi$$

for different #'s a.

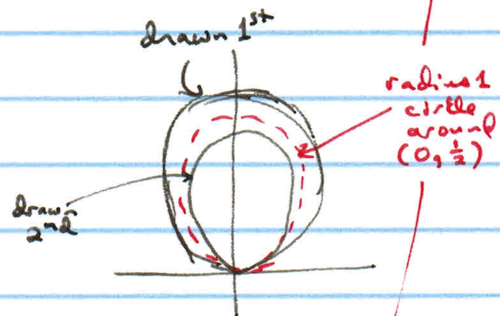
$a=0$

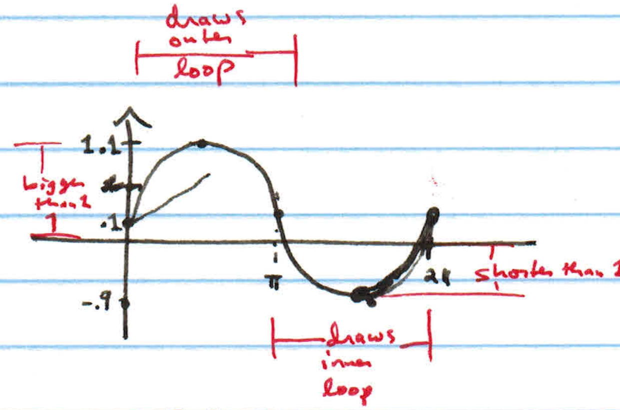




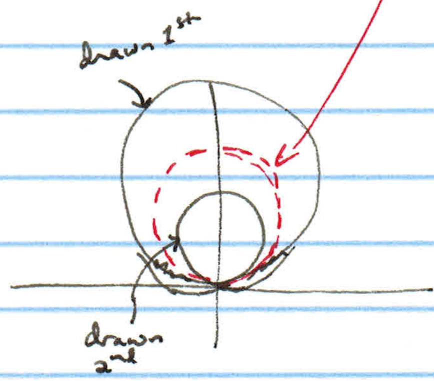
this is the circle of radius 1 around $(0, \frac{1}{2})$, ~~graphed~~ traced twice

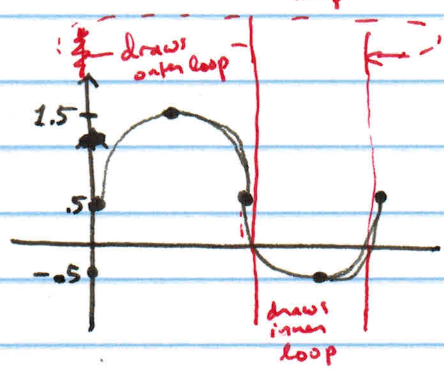
$a=.1$



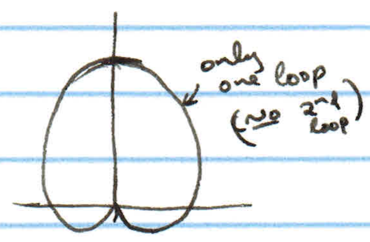


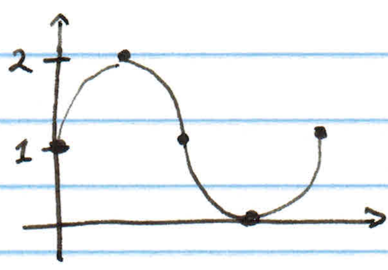
$a=.5$



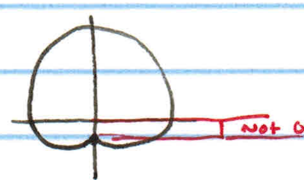


$a=1$





$a=1.1$



$a=5$