Spr 2013

11.8 - Power Series

Remember our goal:

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots$$

(3)

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x \text{ in } (-\infty, \infty)}$$

A power series is a series with a variable to a higher and higher power

Define: A power series is a series

(7) of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$

where each coefficient c_i is a constant #.

Idea: pretend x is a # that isn't revealed till the end

Eg: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$

is a power series

By Geometric series test:

This converges for $|x| < 1$ & diverges for $|x| \geq 1$ } by geometric series test

converges for x in $(-1, 1)$

converges on

-1 0 +1

(5) **Define:** a <u>power series centered at a</u>

is a series of the form [redacted]

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

We will see [redacted] this converges on an interval [redacted]
<u>centered at a</u>

(<u>always</u> converges for $x=a$ because $(a-a)=0$).

---

(5) <u>Eg:</u> [redacted] for what x does

$$\sum_{n=1}^{\infty} \underbrace{\frac{(x-3)^n}{n}}_{a_n} \text{ converge?}$$

THINK:
suppose x is a fixed mystery # [redacted]
that we will be told @ the end of the problem

use the <u>ratio test</u>:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)} \cdot \frac{n}{(x-3)^n}\right| = \cdots$$

we need [redacted] these!

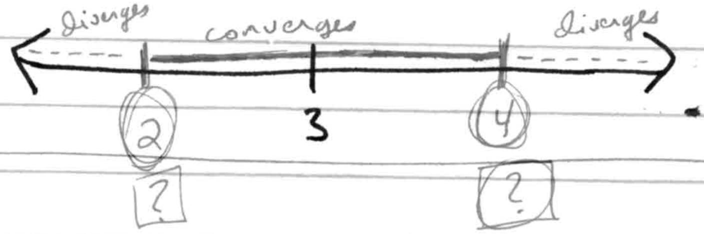
$x-3$ [redacted] can be negative!

$$= |x-3| \cdot \frac{n}{(n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \text{[redacted]}\ |x-3|$$

Draw as we proceed

For what x does the power series converge?



Ratio Test $\Rightarrow$ The series $\boxed{\text{converges}}$ (absolutely)

when $|x-3| < 1$

when $-1 < x-3 < 1$

10 $\boxed{\text{when} \quad 2 < x < 4}$

Ratio Test $\Rightarrow$ The series diverges

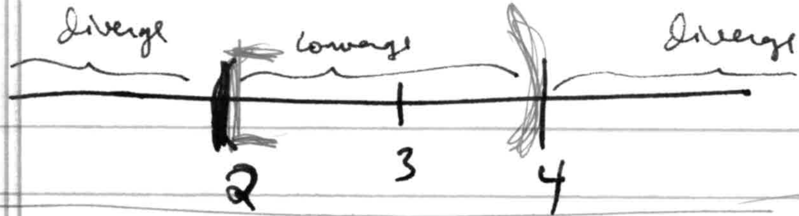
when $|x-3| > 1$

when $x-3 < -1$

or

$x-3 > 1$

$\boxed{\text{when } x < 2 \text{ or } x > 4}$



Same Drawing continued

What about when $|x-3| = 1$?

$$x - 3 = 1 \Rightarrow x = 4$$

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges p-series $p=1$

(10)

$$x - 3 = -1 \Rightarrow x = 2$$

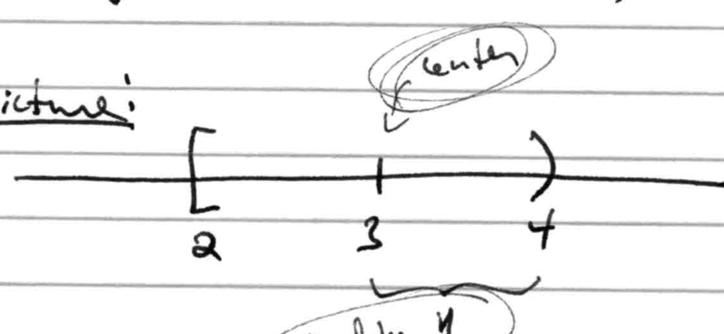
$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ converges alternating series test

<u>Summary:</u>

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$ converges

if x is in $[2, 4)$

<u>picture:</u>



Define: the Radius of convergence
$= \frac{1}{2}$ the length of the interval of convergence

the Ratio/Root test argument gives...

Theorem: for any power series $\sum c_n (x-a)^n$

either ① the series ONLY converges for $x = a$

Radius of convergence $= 0$

or ② the series converges for all x

Radius of convergence $= \infty$

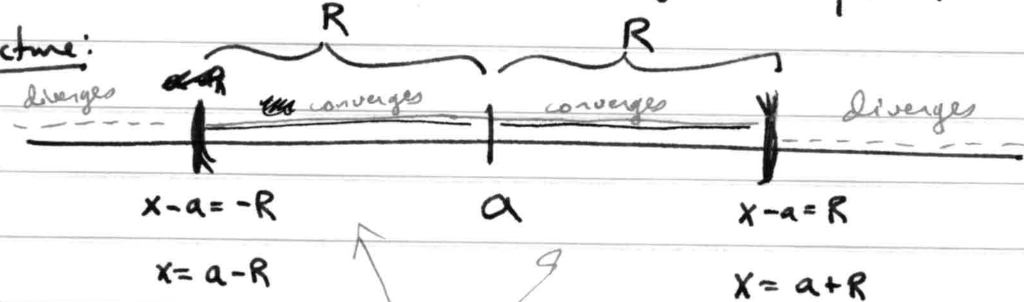
③ there is a number # $R > 0$ so that

series converges if $|x-a| < R$

series diverges if $|x-a| > R$

Radius of convergence is R

Picture:



the interval of convergence is always symmetric around a

(10)

Behavior @ endpoints
Need NOT be symmetric

More Examples Changing The Index:

Eg #1:

$$\sum_{n=0}^{\infty} \frac{n(n+1)\,x^{n}}{(n+2)!}$$

(let $m = n+1$)

(5)

$$= \sum_{m=1}^{\infty} \frac{(m-1)\,m\,x^{m-1}}{(m+1)!}$$

(let $l = n+2$)

$$= \sum_{l=2}^{\infty} \frac{(l-2)(l-1)\,x^{l-2}}{l!}$$

Eg #2:

$$\sum_{n=0}^{\infty} \frac{2^{n+3}\,x^{n+5}}{n^{2}+n\,\text{[redacted]}}$$

(let $m = n+5$)

$$= \sum_{m=5}^{\infty} \frac{2^{m-2}\cdot x^{m}}{\text{[redacted]}(m-5)^{2}+(m-5)\,\text{[redacted]}}$$

[redacted]

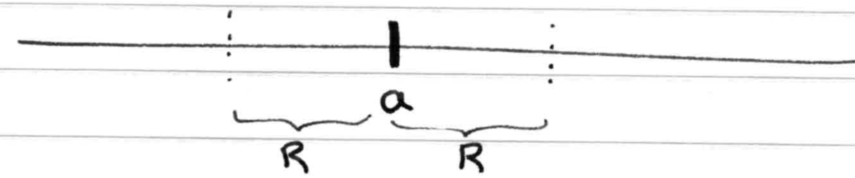
$n+3 = m-2$

$n = m-5$

Power Series II

Last time: any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$

converges on an interval centered at a



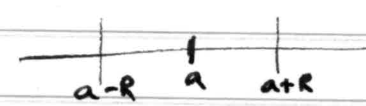
3 radius options: $R=0$ (only converges at $x=a$)

$R=\infty$ (converges for all x)

$R>0$ is a #

Usually, find interval & radius of convergence by

① Ratio/root test to find interval

② draw picture 

③ plug in endpoints to find (or [
) or]

Eg: What is the interval and radius [redacted] radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

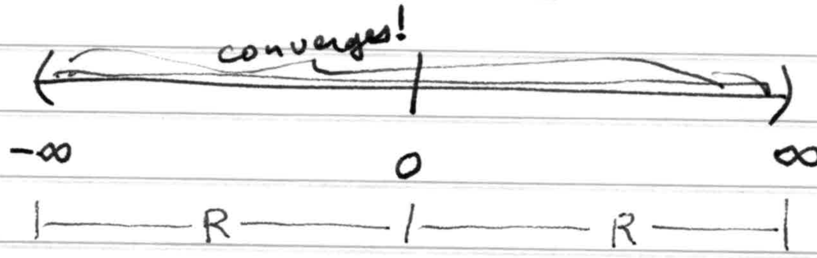
① Root Test

$$|a_n| = \left|\frac{x^n}{n^n}\right| = \left(\left|\frac{x}{n}\right|\right)^n = \left(\frac{|x|}{n}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(\frac{|x|}{n}\right)^n\right)^{\frac{1}{n}} = \frac{|x|}{n} = 0 < 1$$

$\boxed{\text{this is FOR ALL } x}$

Root test $\Rightarrow$ the series converges for every x

② 

③ No endpoints to plug in!

The interval of convergence is $(-\infty, \infty)$

The radius of convergence $= \infty$

Sometimes, we can find the interval of convergence $\boxed{\text{more quickly}}$

Eg: find the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$ $\boxed{\text{Power series centered at}}$ $\boxed{-1}$

NOTICE: this is geometric.

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\frac{(x+1)^2}{9}\right)^n$$

This converges $\Leftrightarrow \left|\frac{(x+1)^2}{9}\right| < 1$

$$\Leftrightarrow \left|(x+1)^2\right| < 9$$

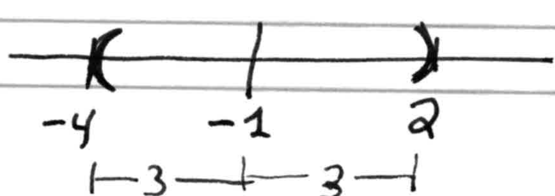
$$\Leftrightarrow \left(|x+1|\right)^2 < 9$$

$$\Leftrightarrow |x+1| < 3$$

$$\Leftrightarrow -3 < (x+1) < 3$$

$$\Leftrightarrow -4 < x < 2$$

Interval of convergence:



a.k.a. $(-4, 2)$

Extra Example
Skip in class
to Return exams

Eg: What is the interval and radius of convergence of

$$\sum_{n=0}^{\infty} \boxed{\frac{3^n \cdot \ln(n+1)}{\sqrt{n+1}}} \cdot x^n \quad \Big\} \quad \boxed{\text{a power series centered at } 0}$$

the constant c_n
(doesn't depend on x).

① Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{3^{n+1} \cdot \ln(n+2) \cdot x^{n+1}}{\sqrt{n+2}} \right)}{\left(\frac{3^n \cdot \ln(n+1) \cdot x^n}{\sqrt{n+1}} \right)} \right|$$

[Now very important]

$$= \lim_{n\to\infty} \left| \underbrace{\frac{3^{n+1}}{3^n}}_{=3} \cdot \frac{\ln(n+2)}{\ln(n+1)} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot \underbrace{\frac{x^{n+1}}{x^n}}_{=x} \right|$$

$$\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(n+1)} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \ldots = 1$$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = \ldots = 1$$

$$= 3 \cdot |x|$$

Ratio Test

$\Rightarrow$ the series converges for

$$3 \cdot |x| < 1$$

$\Rightarrow$ converges for

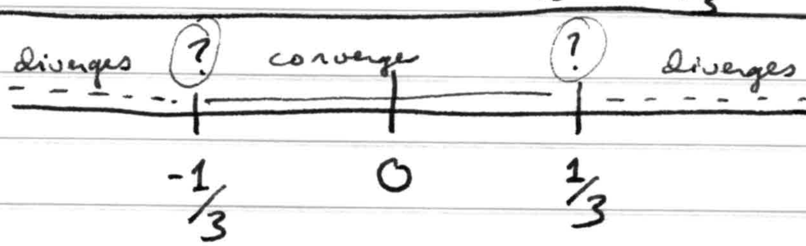
$$-\tfrac{1}{3} < x < \tfrac{1}{3}$$

Ratio Test

$\Rightarrow$ diverges for $\quad x > \tfrac{1}{3}$

$\qquad\qquad\qquad\qquad x < -\tfrac{1}{3}$

Draw Picture:



these cancel
equality 1

when $x = \tfrac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{3^n \cdot \ln(n+1) \cdot x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{3^n \cdot \ln(n+1) \left(\frac{1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{\ln(n+1)}{\sqrt{n+1}}$$

NOTICE: $\quad \dfrac{\ln(n+1)}{\sqrt{n+1}} \geq \dfrac{1}{\sqrt{n+1}}$

& $\sum \frac{1}{\sqrt{n+1}}$ diverges by limit comparison to $\sum \frac{1}{\sqrt{n}}$

the series diverges by direct comp to $\sum \frac{1}{\sqrt{n+1}}$

when $x = \frac{-1}{3}$

Then $\displaystyle\sum_{n=0}^{\infty} \frac{(3)^n \cdot \ln(n+1)\, x^n}{\sqrt{n+1}}$

$\boxed{\begin{array}{l} \underline{\text{NOTICE}} \\ 3^n \cdot \left(-\frac{1}{3}\right)^n = \left(\frac{-3}{3}\right)^n \\ \qquad\qquad = (-1)^n \end{array}}$

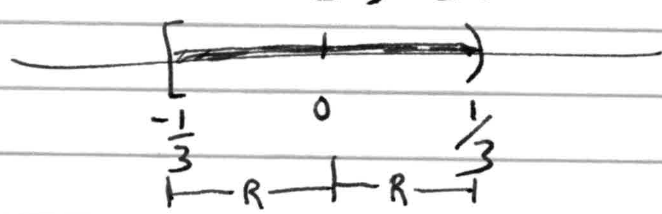
$$= \sum_{n=0}^{\infty} \frac{3^n \cdot \ln(n+1) \; \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{\ln(n+1)}{\sqrt{n+1}} \cdot (-1)^n$$

<u>Notice:</u>
- alternating
- $\frac{\ln(n+1)}{\sqrt{n+1}}$ decreases to 0

converges by alternating series test.

<u>So:</u> The ~~[illegible]~~ Interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right)$



The radius of convergence is $\frac{1}{3}$.

Represent Functions as Power Series

[redacted]

Yesterday: for what x does $\sum_{n=0}^{\infty} c_n x^n$ converge?

Today: What does $\sum_{n=0}^{\infty} c_n x^n$ equal

[redacted]

Notice: Power series are functions.

(5) plug in a # from interval of convergence

get out the # the series converges to

We will flip this:

Given a function

We want to find a power series for it

Prototypical Example

$$\sum_{n=0}^{\infty} x^n = \underbrace{1 + x + x^2 + \cdots}_{\text{geometric}} = \frac{1}{1-x} \quad \text{when } |x| < 1$$

geometric

first term = 1

ratio = x

(5) **So** flipping gives:

$$f(x) = \underbrace{\frac{1}{1-x}}_{\text{the function}} = \underbrace{\sum_{n=0}^{\infty} x^n}_{\text{the representation}} \quad \underbrace{\text{for } |x| < 1}_{\text{where the representation is VALID}}$$

on the interval $(-1, 1)$

This, plus algebra, gives many power series representations for FREE

Eg: let $f(x) = \dfrac{2x}{1+x^2}$

find ① a power series for f (a power series to <u>represent</u> f)

② where the series converges

$$f(x) = \frac{2x}{1+x^2}$$

(could represent: $\dfrac{1}{1-\square}$

<u>Trick ①</u>: pull $2x$ off top

<u>Trick ②</u>: $1+x^2 = 1-(-x^2)$)

(10) $$= 2x \cdot \left(\frac{1}{1+x^2}\right)$$

$$= 2x \cdot \left(\frac{1}{1-\boxed{(-x^2)}}\right)$$

<u>Know:</u> $\dfrac{1}{1-\square} = \displaystyle\sum_{n=0}^{\infty} \square^n$

$$= 2x \cdot \sum_{n=0}^{\infty} \boxed{(-x^2)}^n$$

$$= 2x \cdot \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

Because $2x$ is constant with respect to the sum:

$$= \sum_{n=0}^{\infty} 2 \cdot (-1)^n \cdot x^{2n+1}$$ ← (from the x outside the sum)

[Do we re-index or leave this alone?

Two factors: ① aesthetics

② what does Web-Assign / the Exam ask for?

This is nice & clean, and we are not given a starting index

⇒ leave it alone]

for what x does this converge?

Notice: convergence depends on the geometric sum

converges $\Leftrightarrow$ $|-x^2| < 1$

(3) $\Leftrightarrow$ $|x|^2 < 1$

$\Leftrightarrow$ $-1 < x < 1$

converges for x in $(-1, 1)$

Eg: let $f(x) = \frac{1}{25 - x^2}$

find ① power series & ② where it converges

$$f(x) = \frac{1}{25 - x^2} \quad \left(\text{can represent } \frac{1}{1 - \square} \right)$$

$$= \frac{1}{25} \cdot \frac{1}{1 - \left(\boxed{\frac{x^2}{25}}\right)}$$

(7)

move quick

skip steps if needed

$$= \frac{1}{25} \cdot \sum_{n=0}^{\infty} \left(\frac{x^2}{25} \right)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{25^{n+1}}$$

convergence depends on geometric sum part

$$\text{Converges} \iff \left| \frac{x^2}{25} \right| < 1$$

$$\iff \left| \frac{x}{5} \right|^2 < 1$$

$$\iff -1 < \frac{x}{5} < 1$$

$$\iff -5 < x < 5$$

converges for x in $(-5, 5)$

We can also do Calculus with power series

Theorem: (You take the derivative term-by-term)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n \cdot x^n\right]$$

(5)

$$= \frac{d}{dx}\left[c_0 + c_1\cdot x + c_2\cdot x^2 + \cdots\right]$$

$$= \left(\text{No } c_0 \text{ term}\right) + 1\cdot c_1\cdot x^0 + \left(2\cdot c_2\cdot x^1\right) + \left(3\cdot c_3\cdot x^2\right) + \cdots$$

$$= \sum_{n=1}^{\infty} n\cdot c_n\cdot x^{n-1}$$

Should we re-index?

consider ① Aesthetics

or ② desired answer

lets leave it for now, & do on case-by case basis

Re-stated Theorem: (You take the derivative term-by-term)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n \cdot (x-a)^n\right] = \sum_{n=1}^{\infty} \frac{d}{dx}\left[c_n \cdot (x-a)^n\right]$$

What happens to convergence?

① Radius of convergence doesn't change

② Endpoints might change.

Eg: $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n\right]$$

$$= \sum_{n=1}^{\infty} \frac{d}{dx}\left[\boxed{\frac{2^n}{n!}}\, x^n\right]$$

constant with respect to x

$$= \sum_{n=1}^{\infty} \left(\frac{2^n}{n!}\right) \cdot \frac{d}{dx}[x^n]$$

(10)

$$= \sum_{n=1}^{\infty} \left(\frac{2^n}{n!}\right)\left(n \cdot x^{n-1}\right)$$

not pretty yet!

first: simplify

$$= \sum_{n=1}^{\infty} 2^n \cdot \frac{n}{n!} \cdot x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{(n-1)!} x^{n-1}$$

next: re-index. $m = n-1$

$$= \sum_{m=0}^{\infty} \frac{2^{m+1}}{m!} x^m$$

finally: re-name

original derivative $$= \sum_{n=0}^{\infty} \frac{2^{n+1}}{n!} x^n$$ ← very pretty!

$\underline{\text{Eg:}}$ $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{2^n}{n^2+5} x^{2n+1}\right]$$

$$= \sum_{n=1}^{\infty} \frac{d}{dx}\left[\underbrace{\frac{2^n}{n^2+5}}_{\text{constant WRT } x} \cdot x^{2n+1}\right]$$

$$\boxed{= \sum_{n=1}^{\infty} \frac{2^n}{n^2+5} \cdot (2n+1) \cdot x^{2n}}$$

(re-indexing doesn't really improve this)

Represent Functions as Power Series II

Last time: take derivatives term-by-term

This time: Integrate term-by-term

even better: we get...

Theorem: $$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n \right] dx = \int \left[c_0 + c_1(x-a)^1 + \dots \right] dx$$

one single constant of integration

$$= \boxed{C} + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + \cdots$$

(7)

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

[redacted]

what happens to convergence?

1. Radius of convergence stays the same
2. Endpoints can change

(3)

write quick!

We can use series calculus to

① to find power series reps for more functions

② to approximate the area under tricky functions

Remember!

(10)-(15)

Eg: find a power series rep for $\ln(1+x)$ and its radius of convergence

$$\ln(1+x) = \int \frac{1}{1+x}\, dx$$

$$= \int \frac{1}{1-(-x)}\, dx = \int \left[\sum_{n=0}^{\infty} (-x)^n \right] dx$$

$$= \int \left[1 - x + x^2 - x^3 + \cdots \right] dx$$

$$= C + x \text{ [redacted] } - \frac{x^2}{2} + \frac{x^3}{3} \text{ [redacted] } - \ldots$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \text{ [redacted] } \frac{x^{(n+1)}}{n+1} \text{ [redacted]}$$

to find C, plug in $x=0$ to both sides
(this kills all x terms in the sum)

$$\ln(1+0) = C + 0$$

$$\Rightarrow C = 0$$

so:

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}$$

Re-indexing gives

Radius of Convergence:

$\sum_{n=0}^{\infty} (-x)^n$ converges with Radius $= 1$

& integrating doesn't change radius

$\boxed{\Rightarrow \text{Radius of convergence} = 1}$

Eg: Find power series rep
& radius of convergence
for $\tan^{-1}(x)$

Remember! $\hookrightarrow$

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

(10)

$$= \int \left[\sum_{n=0}^{\infty} (-x^2)^n \right] dx$$

$$= \int \left[\sum_{n=0}^{\infty} \boxed{(-1)^n} \cdot x^{2n} \right] dx$$

doesn't change WRT x

Integrating term-by-term gives

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)}$$

plug in $x=0$ to find C

$\tan^{-1}(0) = C + 0$

$\Rightarrow 0 = C$

$$\boxed{\text{So } \tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)}}$$

with Radius of convergence $= 1$

Radius argument:

① $\sum(-x^2)^n$ converges for $|x|<1$

② integrating doesn't change radius

(Skip if less than 20 min left)

Optional aside:

we don't have the tools to prove it

but it is true that

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)}$$

Because $\tan^{-1}(1) = \frac{\pi}{4}$,

we learn

$$\pi = \underbrace{\sum_{n=0}^{\infty} 4 \cdot (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)}}$$

this is an alternating series that decreases to 0

so

we can compute digits of π to ANY desired error

(To learn more, see the Microsoft Excel Worksheet on my teaching website)

One payoff for power series is a very nice integration method

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<u>Eg:</u> (A) Evaluate $\int \frac{1}{1+x^7}\,dx$ as a power series

and

(B) Use this to approximate

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx$$

with $|\text{error}| \le 10^{-7}$

---

(10) (A) $\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n}$

$\boxed{\text{So}}$

(B) $$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \int_0^{.5} \Big[\underset{n=0}{1} - \underset{n=1}{x^7} + \underset{n=2}{x^{14}} - x^{21} + \dots\Big]\,dx$$

$$= \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots\right]_0^{\frac{1}{2}}$$

definite integral
⇒ constant cancels

$$= \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} + \dots$$

this is an alternating series

$\Rightarrow |\text{error of } s_n| \le |a_{n+1}|$

Note/compute:

a $\quad \frac{1}{22 \cdot 2^{22}} \approx 1.08 \times 10^{-8}$

(5)

$\boxed{\text{so}}$ the integral is approximately

$$\frac{1}{2} - \frac{1}{8 \cdot 2^{8}} + \frac{1}{15 \cdot 2^{15}}$$

with error $\leq 10^{-7}$

Taylor and Maclaurin Series I

why study power series?

→ can use power series to approximate integrals

→ Functions like e^x, $\ln(x)$, $\sin(x)$, $\cos(x)$, etc give us a rich mathematical language for describing processes

Power series give us a way to compute #'values for these functions

(10)

What if we can represent

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$ for $|x-a| < r$?

then $\boxed{f(a) = c_0} + 0 + \dots$

Also: $f'(x) = c_1 + 2\cdot c_2(x-a) + 3\cdot c_3(x-a)^2 + \dots$

so $\boxed{f'(a) = c_1} + 0 + \dots$

Also: $f''(x) = 2\cdot c_2 + 3\cdot 2\cdot c_3(x-a) + 4\cdot 3\cdot c_4\cdot(x-a)^2 + \dots$

so $\boxed{f''(a) = 2\cdot c_2}$

Also: $f'''(x) = 3\cdot 2\cdot 1\cdot c_3 + 4\cdot 3\cdot 2\cdot c_4\cdot(x-a) + \dots$

so $\boxed{f'''(a) = 3!\cdot c_3}$

In Fact: $f^{(n)}(a) = n!\cdot c_n \Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$

We have proved:

Theorem: If $f(x)$ has a power series representation around a with radius $R>0$

Then $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and the series has radius of convergence $= R$

(10) this is a very useful series, so we give it a name!

Define: the Taylor series of f around a is the series $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the case of series centered at 0 (with $a=0$) gets its own special name:

Define: the Maclaurin series of f is the series $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

taylor series centered around 0

Eg: find the Maclaurin series for $f(x) = e^x$

Notice: $f'(x) = e^x$

$f^{(n)}(x) = e^x$ for all n

so $f^{(n)}(0) = e^0 = 1$ for all n

(10) so
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!}$$
$$= 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

the Maclaurin series for e^x

Ratio test: this series converges for all x

Mysterious Fact: e^x has a power series rep around 0

(we'll come back to this in a couple days)

$\Rightarrow$ by the theorem

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x}$$

(5)

Eg: find the taylor series for $f(x) = e^x$ around $a = 2$

Again: $f'(x) = e^x$

$f^{(n)}(x) = e^x$ for all n

So $f^{(n)}(2) = \boxed{e^2}$ for all n

$$\boxed{\sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(2)}}{n!}(x-2)^n} = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

(the Taylor series formula around 2)

(centered around 0)

Eg: find the Maclaurin series for $f(x) = \sin(x)$

Goal: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

It is helpful to use a table to keep track of computations:

(10)

n	$f^{(n)}(x)$	f
0	$\sin(x)$	$\sin(0) = 0$
1	$\cos(x)$	$\cos(0) = 1$
2	$-\sin(x)$	$-\sin(0) = 0$
3	$-\cos(x)$	$-\cos(0) = -1$
4	$\sin(x)$	(repeats!)

So:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + 1 \cdot x + \frac{0 \cdot x^2}{2!} + \frac{(-1)x^3}{3!} + \cdots$$

coefficients repeat.

$$= \underbrace{x - \frac{x^3}{3!}}_{\text{first 4 terms}} + \underbrace{\frac{x^5}{5!} - \frac{x^7}{7!}}_{\text{Next 4 terms}} + \cdots$$

think back to sequences... play with the series... and get

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Ratio test: this series converges for **all** x

(for x in $(-\infty, \infty)$)

Fact: $\sin(x)$ has a power series representation around 0

(5)

$\Rightarrow$ By theorem:

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \text{ in } (-\infty, \infty)}$$

Taylor & Maclaurin Series II

Eg: Find the Maclaurin Series for $f(x) = (1+x)^{\pi}$

Make a table:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{\pi}$	$1^{\pi} = 1$
1	$\pi(1+x)^{\pi-1}$	$\pi \cdot (1+0)^{\pi-1} = \pi \cdot 1 = \pi$
2	$\pi(\pi-1)(1+x)^{\pi-2}$	$\pi\cdot(\pi-1)\cdot(1+0)^{\pi-2} = \pi(\pi-1)$
3	$\pi(\pi-1)(\pi-2)(1+x)^{\pi-3}$	$\pi\cdot(\pi-1)\cdot(\pi-2)\cdot(1+0)^{\pi-3} = \underbrace{\pi(\pi-1)(\pi-2)}$

(15)

Describe pattern! third term has π down to $(\pi-2)$

Generalize Pattern! n^{th} term has π down to $(\pi-(n-1))$
$= (\pi-n+1)$

Use formula:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\pi(\pi-1)(\pi-2)\cdots(\pi-n+1)}{n!} x^n$$

the expression on top is messy:

Define: for any # k, $\binom{k}{n} = \dfrac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

this lets us rewrite the series

$$\boxed{\text{The Maclaurin series for } (1+x)^{\pi} = \sum_{n=0}^{\infty} \binom{\pi}{n} x^n}$$

to summarize:

(5-10)

Important Maclaurin Series (memorize!)

$\frac{1}{1-x} =$ 3 days ago

$\ln(1+x) =$ 2 days ago

$\tan^{-1}(x) =$ 2 days ago

Radius of convergence $R = 1$

$e^x =$ yesterday

$\sin(x) =$ yesterday

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Radius of convergence $R = \infty$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$$

Radius of convergence $R = 1$

(by ratio test. see textbook.)

Applications:

Eg: find the Maclaurin Series for $x \cdot \cos(x^3)$

$$\boxed{x} \cdot \cos\left(x^3\right) = \boxed{x} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^3\right)^{2n}}{(2n)!}$$

(10)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}$$

Eg: Evaluate $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n \cdot 2^{2n}}$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \cdot \frac{3^n}{2^{2n}}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{4}\right)^n}{n}$$

$$= \ln\left(1 + \tfrac{3}{4}\right) = \underline{\ln\left(\tfrac{7}{4}\right)}$$

In fact, we can now get a #-value for the alternating harmonic series!

Eg: Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

endpoints of interval of convergence are tricky... but this does work here

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n} = \ln(1+1) = \ln(2)$$

(15)

Eg: (A) Evaluate $\int e^{-x^2}\,dx$ as a power series.

(B) Approximate $\int_0^1 e^{-x^2}\,dx$ with $|\text{error}| \le .1$

(A) Remember: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so: $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$

$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

Integrating, we get: $\int e^{-x^2}\,dx = \int \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$

$= C + x - \frac{x^3}{3\cdot(1!)} + \frac{x^5}{5\cdot(2!)} - \frac{x^7}{7\cdot(3!)} + \cdots$

(B) (As a definite integral, the constant goes away) $\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right]_0^1$

(the lower bound yields all zero terms so...) $= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots$

$= 1 - \frac{1}{3} + \boxed{\frac{1}{10}} - \frac{1}{42} + \cdots$

$b_{n+1} = .1 = \frac{1}{10}$

By Alternating error estimate, the area is around $1 - \frac{1}{3}$ with $|\text{error}| \le \frac{1}{10}$

An example from pg 129 of
<u>Engineering Statistics</u>, 4^{th} edition
by Montgomery, Runger, & Hubel (2007)

If you make Resistors
where the mean resistance is $100\,\Omega$
and the standard deviation is $10\,\Omega$

(if time) Then

$$\left(\begin{array}{l}\text{The probability}\\ \text{that 25 randomly}\\ \text{chosen resistors have}\\ \text{average resistance} < 95\,\Omega\end{array}\right) = \int_{-\infty}^{\left(\frac{95-100}{\left(\frac{10}{\sqrt{25}}\right)}\right)} \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-u^2}{2}}\,du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2.5} e^{\frac{-u^2}{2}}\,du$$

Theorem: If the derivatives of $f(x)$ are bounded on the interval $(a-R, a+R)$

Then $f(x)$ has a power series representation around a on the interval $(a-R, a+R)$

(if time)

Eg: $\sin(x)$ has a power series representation around 0 on $(-\infty, \infty)$

proof: $\left|\frac{d^n}{dx^n}\sin(x)\right|$ is either $|\sin(x)|$ or $|\cos(x)|$

Both these functions are bounded by 1 on $(-\infty, \infty)$

Therefore $\sin(x)$ has the power series representation around 0 on $(-\infty, \infty)$ $\blacksquare$

Eg: e^x has a power series representation around 0 on $(-\infty, \infty)$

Two step proof:

① for each $R > 0$, $\frac{d^n}{dx^n}(e^x) = e^x$ is bounded by R on $(-R, R)$

so e^x has a power series rep around 0 on each interval $(-R, R)$

② These power series rep's are all the same by the derivation of the Taylor series formula $\blacksquare$

Approximating functions with Taylor Polynomials I

The last few days:

we've looked at power series that REPRESENT functions

Today:

we look at polynomials that APPROXIMATE functions

Define: The n^{th} degree Taylor polynomial of $f(x)$ around a

is

(10) The n^{th} partial sum [redacted]

of the Taylor series of $f(x)$ around a

(it is the sum of the 0^{th} term - to - the - n^{th} term)

said another way...

Define: The n^{th} degree Taylor polynomial of $f(x)$ around a

is $$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notational warning: [redacted]

sometimes [redacted] you will see things like [redacted] $T_3(x) = T_4(x)$

This can happen [redacted] if the taylor series has no even powers [redacted]

(you will see this happen on the written homework)

Key idea: Taylor polynomials around a are very good approximations near a

---

Different polynomials are better at approximating e^x for different x

Eg: let $f(x) = e^x$

around $a=0$: $T_2(x) = e^0 + e^0 \cdot x + \frac{e^0}{2!}x^2$

$$= 1 + x + \frac{x^2}{2}$$

close to e^x for x near 0

(10-15)

around $a=1$: $T_2(x) = e^1 + e^1(x-1) + \frac{e^1}{2!}(x-1)^2$

$$= e + e(x-1) + \frac{e}{2}(x-1)^2$$

$$= \cdots = \frac{e}{2} + \frac{e}{2}x^2$$

close to e^x for x near 1

Higher degrees give better approximations

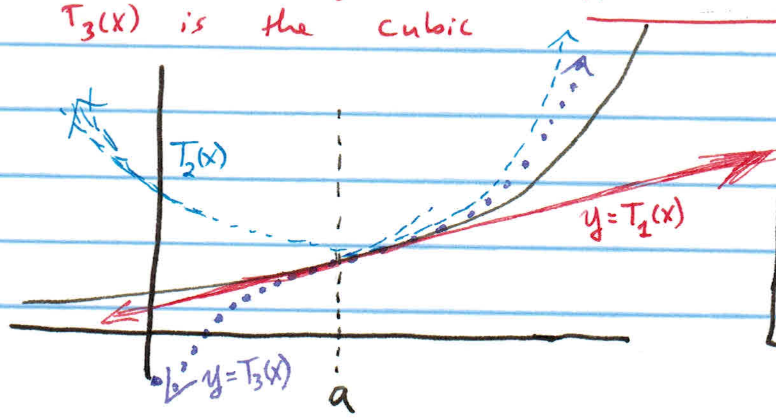
Eg: The first Taylor polynomial around a is

$$T_1(x) = f(a) + f'(a)(x-a)$$

which is the tangent line to $f(x)$ at a

$T_2(x)$ is the quadratic approximation to $f(x)$ at a

$T_3(x)$ is the cubic approximation to $f(x)$ at a



The bigger n is the better an approx $T_n(x)$ gives to $f(x)$ near a

How Good is an approximation?

Define: $R_n(x) = \underset{T_n(x)}{\text{error of}} = f(x) - T_n(x)$

Idea:

$$R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$$

Taylor's Inequality: (error estimate)

(10-15) **If** $\left|f^{(n+1)}(x)\right| \leq M$

for all x in $[a-d, a+d]$

Then $\left|R_n(x)\right| = \left|\text{error of } T_n(x)\right| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

for all x in $[a-d, a+d]$

Eg: All derivatives of $\sin(x)$

have $\left|\frac{d^n}{dx^n}\sin(x)\right| \leq 1$ for all x

So for any center a

and any degree n

and any # x

$$\left|\text{error of } T_n(x)\right| \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

Going further from a drops the accuracy a lot

But increasing n increases the accuracy a TON

Eg: let $f(x) = e^x$

Recall around 0, $T_3(x) = 1 + x + \frac{x^2}{2}$

(A) How accurate is $T_3(x)$ for numbers in $[-2,2]$?

(B) How about for #'s in $[-1,1]$?

Remember: $R_3(x) \le \frac{f^{(3+1)}(0)}{(3+1)!}(x-0)^{3+1}$

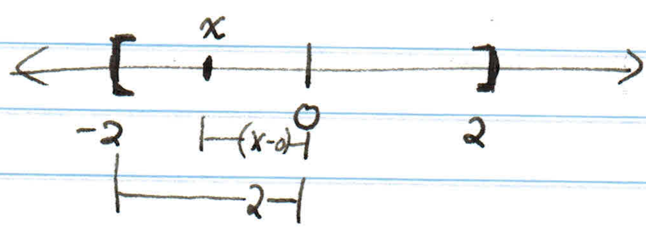
(A) Notice because e^x is increasing on $[-2,2]$ and because $f^{(4)}(x) = e^x$

That $f^{(4)}(x) \le e^2$ for all x in $[-2,2]$

<u>So</u>: Taylor's Inequality says

$$\left|\begin{matrix}\text{error of } T_3(x) \\ \text{on } (-2,2)\end{matrix}\right| = |R_3(x)| = \frac{e^2}{4!}(x-0)^4$$

<u>Ask</u>: what is the maximum distance $|x-0|$ in $[-2,2]$?



<u>Notice</u>: $|x-0| \le 2$ for all x in $[-2,2]$

<u>conclude</u>:

$$\left|\begin{matrix}\text{error of } T_3(x) \\ \text{on } [-2,2]\end{matrix}\right| \le \frac{e^2}{4!}|2|^4 \approx 4.89$$

that's a big error!

(B) same computation gives

$$\left|\begin{matrix}\text{error of } T_3(x) \\ \text{on } [-1,1]\end{matrix}\right| \le \frac{e}{4!}|1|^4 \approx .113$$

this is a small error!

Approximating functions with Taylor Polynomials II.

Remember: the n^{th} degree Taylor Polynomial is the n^{th} partial sum of the Taylor Series

It is the sum of the 0^{th} through the n^{th} terms of the Taylor series.

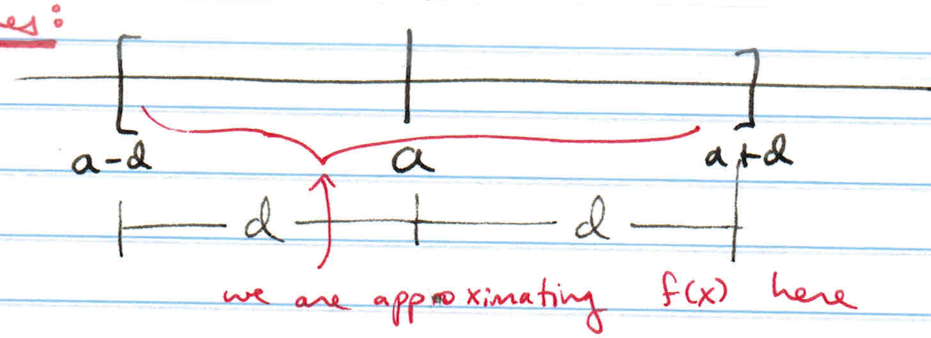
Remember: The n^{th} degree Taylor polynomial around a

(5) is $T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$R_n(x) = \text{Error of } T_n(x) = f(x) - T_n(x)$

IDEA: for x near a, $T_n(x)$ approximates $f(x)$

In pictures:



happily, we can estimate the error of our approximation:

Taylor's Inequality:

Some #

If $\left|f^{(n+1)}(x)\right| \le M$ for all $|x-a| \le d$

Then $\left|\begin{matrix}\text{error of}\\ T_n(x)\end{matrix}\right| = |R_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$

Notice: this looks like the biggest term in $R_n(x)$

(5-7) for all $|x-a| \le d$

This is not an accident!
the biggest term of the tail of the Taylor series is its most important term

In words:

If $f^{(n+1)}(x)$ is bounded by M for all x in $[a-d, a+d]$

Then $\left|\begin{matrix}\text{error of}\\ T_n(x)\end{matrix}\right| \le \dfrac{M}{(n+1)!}|x-a|^{n}$

for all x in $[a-d, a+d]$.

Eg: Find the 3^{rd} degree Taylor polynomial for $f(x) = \ln(x)$ around a

$$T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

To simplify $T_n(x)$, we must evaluate $f(a) = \ln(a)$ <u>and</u> $f^{(n)}(a)$ for each n.

(10)

Eg: Find 3^{rd} degree taylor polynomials for $\ln(x)$ around $a = 1$ and around $a = e$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(e)$
0	$\ln(x)$	$\ln(1) = 0$	$\ln(e) = 1$
1	$\frac{1}{x}$	$\frac{1}{1} = 1$	$\frac{1}{e}$
2	$\frac{-1}{x^2}$	$\frac{-1}{1^2} = -1$	$\frac{-1}{e^2}$
3	$\frac{2}{x^3}$	$\frac{2}{1^3} = 2$	$\frac{2}{e^3}$
(No more terms needed!)			

<u>Around $a = 1$:</u>

$$T_3(x) = 0 + \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$\frac{2}{6} = \frac{1}{3}$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

Around $a = e$:

$$T_3(x) = 1 + \frac{\frac{1}{e}}{1!}(x-e) + \frac{\frac{-1}{e^2}}{2!}(x-e)^2 + \frac{\frac{2}{e^3}}{3!}(x-3)^3 \qquad \frac{\frac{2}{e^3}}{3!} = \frac{2}{6\cdot e^3}$$

$$= 1 + \frac{1}{e}(x-e) - \frac{1}{2\cdot e^2}(x-e)^2 + \frac{1}{3\cdot e^3}(x-e)^3$$

(5)

This gives us two DIFFERENT ways to approximate $\ln(x)$

It is natural to ask which is better for approximating $\ln(\dots)$ for different #'s.

We said before that

$$\ln(2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

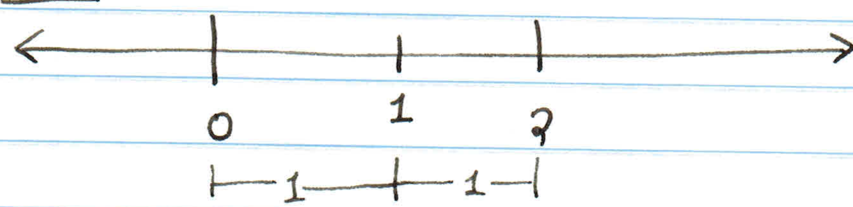
using alternating error, we can approximate $\ln(2)$

BUT alternating error usually takes a LOT more terms than Taylor's error estimate.

Eg! [redacted]

Can we use Taylor's inequality and $T_3(x)$ centered at $a=1$ to approximate $\ln(2)$?

NOTICE:



(10)

so $|2-1| = |x-a| = d = 1$

Ask: is $|f^{(3+1)}(x)| = |f^{(4)}(x)| = \left|\frac{-6}{x^4}\right|$

bounded for $|x-1| \le 1$

That is: is $\frac{6}{x^4}$ bounded for x in $[0,2]$?

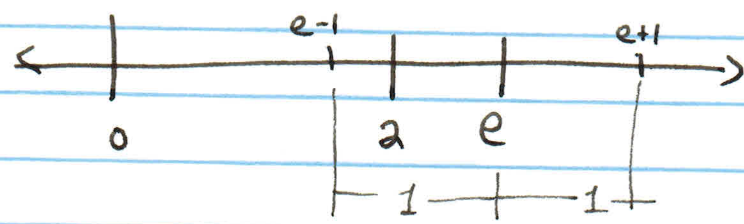
Answer: NO!

so $T_3(x)$ centered at $a=1$ will NOT approximate $\ln(2)$ well.

Eg: Can we use Taylor's inequality and $T_3(x)$ centered at $a = e$ to approximate $\ln(2)$

(5)

NOTICE:



Ask: Is $\left|f^{(4)}(x)\right| = \dfrac{6}{x^4}$ bounded for $|x - e| \le 1$

i.e. is $\dfrac{6}{x^4}$ bounded for x in $[e-1, e+1]$?

Answer: yes! by $\dfrac{6}{(e-1)^4}$

By Taylor's Inequality,

Because $\left| f^{(3+1)}(x) \right| \leq \frac{6}{(e-1)^4}$ for x in $[e-1, e+1]$

bound on $(n+1)^{st}$ derivative.

x   a

we know $\left| \begin{array}{c} \text{error of} \\ T_3(2) \end{array} \right| \leq \frac{\boxed{\frac{6}{(e-1)^4}}}{4!} |2-e|^4$

$$\leq \frac{6}{4!} \cdot \frac{1}{(e-1)^4} |2-e|^4$$

(10)

$$\leq \frac{1}{4} \cdot \left| \frac{2-e}{e-1} \right|^4 \approx .0076$$

<u>So</u>

around $a = e$

$$\ln(2) \approx T_3(2) = 1 + \frac{2-e}{e} - \frac{(2-e)^2}{2 \cdot e^2} + \frac{(2-3)^3}{3 \cdot e^3}$$

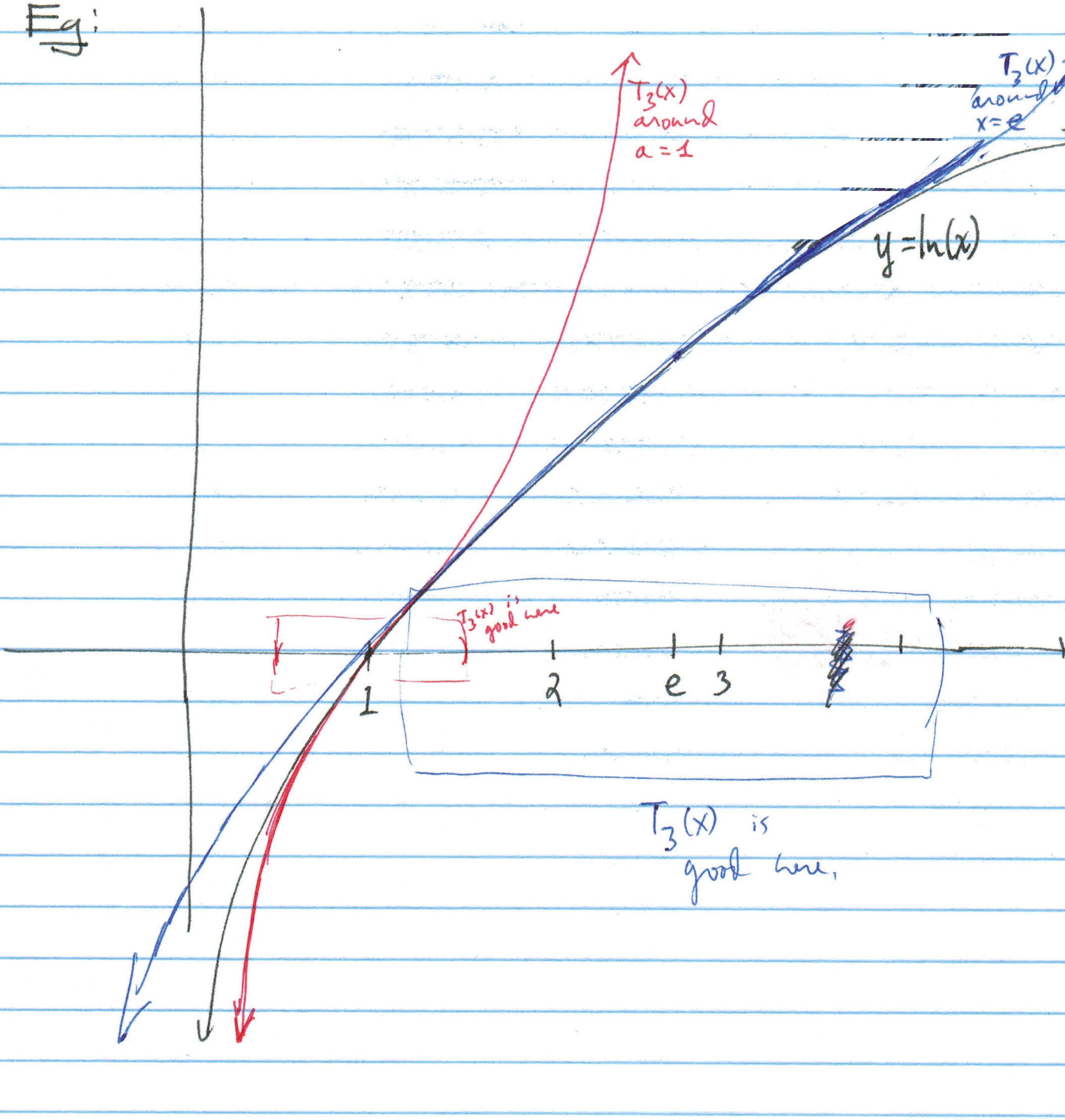
$$\approx .6947$$

(for comparison, calculator gives $\ln(2) \approx .6931$)

Graphing taylor poly's alongside f(x)

~~[crossed out]~~ gives a good picture of how they work as approximations.

Eg:



$y = \ln(x)$ the function being approximated

$y = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ T_3(x) centered around a=1

$y = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$ T_3(x) centered around a=*e*