11.3 - Integral Test. I

Why Convergence tests?

→ **NOT** many methods to find the # value of a series

→ **Many** methods to test if a series converges / diverges

Last Time: saw

Divergence test: **If** $\lim_{n\to\infty} a_n \neq 0$ (or diverges)

Then $\sum_{n=1}^{\infty} a_n$ diverges.

Eg: $\sum_{n=1}^{\infty} (-1)^n$ diverges by divergence test.

But this says Nothing if $\lim_{n\to\infty} a_n = 0$

Eg: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ← uses the next test

we will show this today.

Integral Test:

we'll talk about this part NEXT class

If f is $\underbrace{\text{continuous, positive, decreasing}}$ on $[1, \infty)$

AND if $a_n = f(n)$

Then $\left[\sum_{n=1}^{\infty} a_n \text{ converges}\right]$

$\Leftrightarrow$ if and only if

$\left[\int_1^{\infty} f(x)\,dx \text{ converges}\right]$

The basic idea: An integral is a kind of sum (a Reimann sum)

so it makes sense that convergence of sums & integrals lines up

We'll see a more full explanation next class

So: ① $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

② $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty}$ diverges

Series

Eg: Does the 'Sum' $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

Notice: $f(x) = \frac{1}{x^2+1}$ is continuous, positive, decreasing

AND $a_n = \frac{1}{n^2+1} = f(n)$

Can Apply integral test!

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \left[\underbrace{\tan^{-1}(t)}_{\to \pi/2} - \underbrace{\tan^{-1}(1)}_{= \pi/4} \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

the INTEGRAL converges

$\Rightarrow$ the SERIES converges by the Integral Test

must say this!

Eg: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

$f(x) = \frac{1}{x}$ is cont, positive, & decreasing

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \Big[\underbrace{\ln|t|}_{\to\infty} - \underbrace{\ln|1|}_{=0} \Big] = \infty$$

the integral diverges

$\Rightarrow$ the series diverges
by the integral test

check:

Notice: for all $p>0$ that $\frac{1}{x^p}$ is positive, continuous, and decreasing on $[1,\infty)$

(don't write just say)

Eg: For what $p>0$ does
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converge ?}$$

3 cases:

$p=1 \Rightarrow \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t = \infty$

$\Rightarrow$ the integral [redacted] diverges

$0<p<1$ [redacted] $\Rightarrow$ Eg: $p=.9$

$$\int_1^\infty \frac{1}{x^{.9}}\,dx = \lim_{t\to\infty}\int_1^t x^{-.9}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{(-.9+1)}}{-.9+1}\right]_1^t$$

positive

$$= \lim_{t\to\infty}\left[\frac{x^{.1}}{.1}\right]_1^t = \infty$$

$\Rightarrow$ the integral diverges

Eg: $p=1.1$

$p>1 \Rightarrow \int_1^\infty \frac{1}{x^{1.1}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1.1}\,dx$

negative

$$= \lim_{t\to\infty}\left[\frac{x^{(-1.1)+1}}{(-1.1)+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{x^{-.1}}{-.1}\right]_1^t$$

[redacted]

$\Rightarrow$ the integral converges

(see next)

Because the integral converges if $p > 1$
and diverges if $p \leq 1$

$$\boxed{\begin{array}{l}\text{The Series } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \\ \qquad \text{converges if } p > 1 \\ \text{and diverges if } p \leq 1\end{array}}$$

Name:
the p-series test

by the integral test

11.3 - Integral test II

Say → **Goal:** see where the integral test comes from (why these 3 strange conditions on f?)

Recall: a function $f(x)$

is decreasing on an interval if $f'(x) < 0$ for all x

(10) AND is increasing on an interval if $f'(x) > 0$ for all x

---

list of #'s

Define: a sequence $\{a_n\}_{n=1}^{\infty}$

is increasing if $a_n < a_{n+1}$ for all n

decreasing if $a_n > a_{n+1}$ for all n

monotonic if it is EITHER increasing OR decreasing

Eg: the sequence defined by $a_n = \frac{n}{n^2+1}$ is decreasing

Method 1: show $a_n = \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1} = a_{n+1}$

Method 2: define $f(x) = \frac{x}{x^2+1}$

compute $f'(x) = \ldots$

Notice $f'(x) < 0 \Rightarrow f(x)$ is a decreasing function

SO: $a_n = f(n) > f(n+1) = a_{n+1}$

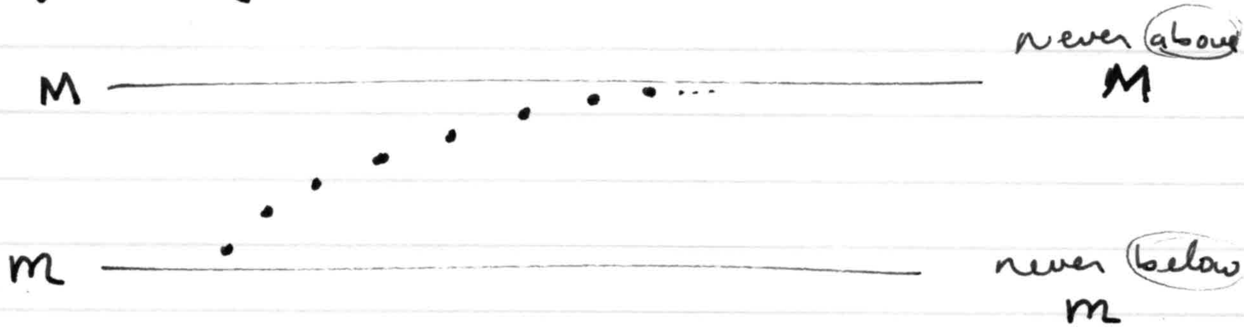
Define: a sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded**

if there are #'s M and m

such that $m \leq a_n \leq M$

for every n

(5) **Eg:**



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list of #'s

Monotonic Sequence Theorem

Every monotonic, bounded sequence

converges (nth term a_n goes to a real #)

(this is a sort of "convergence test" for sequences)

(so we can integrate it)

we need these for the proof

<u>Integral test:</u>

If $f(x)$ is continuous, positive, & decreasing on $[1, \infty)$

<u>and</u> if $a_n = f(n)$

(2-3) Then $\left[\sum_{n=1}^{\infty} a_n \text{ converges}\right] \Leftrightarrow \left[\int_1^{\infty} f(x)\,dx \text{ converges}\right]$

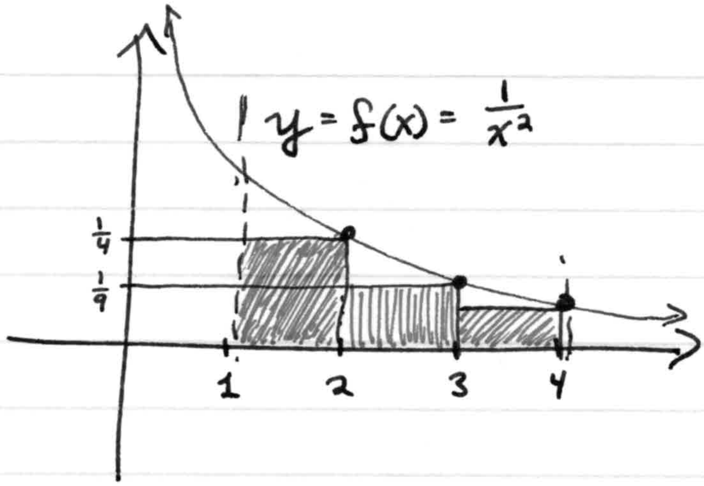
<u>We show:</u> $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(the other direction is similar,
and the choice of $\frac{1}{x^2}$ doesn't change much)

let $a_n = \frac{1}{n^2}$

then $f(x) = \frac{1}{x^2}$

is cont, positive, and decreasing on $[1, \infty)$



(10) **Note:** shaded area $= \frac{1}{2^2} \cdot 1 + \frac{1}{3^2} \cdot 1 + \frac{1}{4^2} \cdot 1 = a_2 + a_3 + a_4$

starts at 2

And: shaded area $\leq \int_1^4 \frac{1}{x^2}\,dx$

BECAUSE f is decreasing and above x-axis.

So: $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} = a_2 + a_3 + a_4 \leq \int_1^4 \frac{1}{x^2}\,dx$

The same idea shows:

$$a_2 + a_3 + \dots + a_n = \sum_{i=2}^{n} a_i \leq \int_1^n f(x)\,dx$$

So: $S_n = \boxed{a_1} + a_2 + a_3 + \dots + a_n \leq \boxed{a_1} + \int_1^n f(x)\,dx$

because f is positive, $\int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx$

so $S_n \leq a_1 + \boxed{\int_1^\infty f(x)\,dx} = a_1 + \boxed{M}$

this is a finite # M Because $\int_1^\infty \frac{1}{x^2}\,dx$ converges

We thus know

$$\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots\}$$
$$= \{s_1, s_2, s_3, \ldots\}$$

is <u>bounded</u>.

Because $f(x)$ is positive,

$$a_n > 0 \text{ for all } n$$

(10) so $\{s_1, s_2, s_3, \ldots\}$ is <u>increasing</u>

By Monotone sequence theorem, the n^{th} term in $\{s_1, s_2, \ldots\}$ converges to a real #.

<u>Therefore</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n\to\infty} s_n$ converges as a series!

this is what we wanted to show.

Just say if low on time

<u>Recap</u>: Because $\int_1^{\infty} \frac{1}{x^2}\,dx$ was finite

<u>AND</u> because this area bounds the increasing sequence of partial sums,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \text{ converges.}$$

We've seen that there is a close connection between Series and Integrals.

It shouldn't be surprising, then, that there is ...

11.4 A Comparison Test for Series

The Comparison Test

(5) Suppose $\sum a_n$ and $\sum b_n$ have positive terms

[redacted]

and [redacted] that $a_n \le b_n$ for all n

(1) if $\sum b_n$ converges then $\sum a_n$ converges — larger converges $\Downarrow$ smaller converges

(2) if $\sum a_n$ diverges then $\sum b_n$ diverges — smaller DIVERGES $\Downarrow$ larger DIVERGES

(This works the same as the comparison test for Integrals.)

Eg: Does this Series Converge?

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$

(compare to something larger)

compare $\frac{n-1}{n^2\sqrt{n}} \leq \frac{n}{n^2\sqrt{n}} = \frac{1}{n^{1.5}}$

(10) and check $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ converges by p-test since $1.5 > 1$

(larger series converges) $\Rightarrow$ the series converges by comparison to $\sum \frac{1}{n^{1.5}}$

Most of the time:

- Compare to a p-series $\left(\sum_{n=1}^{\infty} \frac{1}{n^p}\right)$

(these have quick tests)

or a geometric series $\left(\sum_{n=1}^{\infty} a\cdot r^{n-1}\right)$

- chase down the comparisons that simplify the summand.

Eg: Does this converge?

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$$

$\frac{2^n}{3^n+1} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

and $\sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$ converges, geometric with $|r| = \frac{2}{3} < 1$

(larger series converges)

$\Rightarrow$ the series converges by comparison to $\sum \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$

what if there isn't an obvious simplification,
or
what if the "obvious simplification" doesn't help?

Problematic Example

What does $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ do?

Try DIRECT Comparison

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

a smaller sum

BUT $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

$\Rightarrow$ we learn NOTHING

(instead, we must use)

The Limit Comparison Test

suppose $\sum a_n$ and $\sum b_n$ are series with positive terms

AND that $\lim_{n \to \infty} \frac{a_n}{b_n} = C \neq 0$

a positive, finite #

Then
$$\left[\begin{array}{l} \sum b_n \text{ converges} \Rightarrow \sum a_n \text{ converges} \\ \sum b_n \text{ diverges} \Rightarrow \sum a_n \text{ diverges} \end{array}\right]$$

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges

pick the comparison

we will apply the limit comparison test

with $a_n = \dfrac{1}{2^n - 1}$

$\boxed{b_n = \dfrac{1}{2^n}}$

Because 2^n is clearly the most important term in the denominator

check the Limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(2^n-1)}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{(2^n-1)} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{(2^n-1)} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 \neq 0$$

finite # ✓

check what $\sum b_n$ does

$\boxed{\text{AND}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}}$

geometric series w/ $|r| = \frac{1}{2} < 1$

$\Rightarrow$ converges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges by limit comparison with $\sum \frac{1}{2^n}$

Eg: Does this series converge?

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$$

[often useful trick:
pull out fastest term on top & on bottom]

Limit Comparison

this is the part of a_n that "matters"

$$a_n = \frac{\sqrt{n+2}}{2n^2+n+1} = \underbrace{\frac{\sqrt{n}}{n^2}} \cdot \frac{\sqrt{1+\frac{2}{n}}}{\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}$$

$$b_n = ? = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}$$

check the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{\frac{\sqrt{n}}{n^2}} \cdot \frac{\sqrt{1+\frac{2}{n}}}{\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}}{\cancel{\frac{\sqrt{n}}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n}}}{\left(2+\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{1}{2} \neq 0$$

$0 \quad 0 \quad 0$

↑ finite # $\boxed{\checkmark}$

what does $\sum b_n$ do?

AND

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \text{ converges (p-series, } p>1)$$

$\Rightarrow$ the series converges by limit comparison with $\sum \frac{1}{n^{1.5}}$

Comparison Tests II

Remember the $\boxed{\text{limit comparison test}}$

sums!

If $\sum a_n$ and $\sum b_n$ are series
WITH $a_n > 0$ and $b_n > 0$

AND if $\lim_{n\to\infty} \frac{a_n}{b_n} = C \neq 0$

finite #

(5) Then $\left[\sum a_n \text{ converges} \Leftrightarrow \sum b_n \text{ converges} \right]$.

WARNING:

① a_n & b_n **MUST** be positive

② if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Then you learn NOTHING

~~Does $\sum^{\infty} 3^n + 5^n$~~

(10) Does $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+5^n}{\sqrt{9^n+2^n}}$ converge?

<u>limit Comparison:</u>

$$a_n = \frac{3^n+5^n}{\sqrt{9^n+2^n}} = \frac{5^n\left(\frac{3^n}{5^n}+1\right)}{\sqrt{9^n}\,\sqrt{1+\frac{2^n}{9^n}}}$$

$$= \left(\frac{5}{3}\right)^n \cdot \left(\frac{\left(\frac{3}{5}\right)^n+1}{\sqrt{1+\left(\frac{2}{9}\right)^n}}\right)$$

$\boxed{\text{set}}$ $\displaystyle b_n = \frac{5^n}{\sqrt{9^n}} = \left(\frac{5}{3}\right)^n$

~~Compare~~ compare $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5}{3}\right)^n\left(\frac{\left(\frac{3}{5}\right)^n+1}{\sqrt{1+\left(\frac{2}{9}\right)^n}}\right)}{\left(\frac{5}{3}\right)^n}$

$$= \lim_{n\to\infty} \left(\frac{\overbrace{\left(\frac{3}{5}\right)^n}^{0}+1}{\sqrt{1+\underbrace{\left(\frac{2}{9}\right)^n}_{0}}}\right) = \boxed{1 \neq 0}$$

Check ~~Compare~~ $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n$ diverges.
geometric, $|r| \geq 1$

<u>Conclude</u>: our series diverges by limit comparison with $\sum\left(\frac{5}{3}\right)^n$

Shortcut Method: for a fraction whose top & bottom are increasing (or constant)

① factor out fastest term
from top
and from bottom

② compare to $\dfrac{\text{fastest of top}}{\text{fastest of bottom}}$

(10)

③ & use shortcut to compute limit of ratio

Eg: Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ converge?

$$a_n = \frac{1}{\sqrt[3]{3n^4+1}} = \frac{1}{n^{\frac{4}{3}}} \cdot \frac{1}{\sqrt[3]{3+\frac{1}{n^4}}}$$

Set $\quad b_n = \dfrac{1}{n^{4/3}} = \dfrac{1}{n^{1.33\ldots}}$

Compare: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{3+\frac{1}{n^4}}} = \frac{1}{\sqrt[3]{3}} \neq 0$

$\frac{1}{n^4} \to 0$

Check: $\displaystyle\sum b_n = \sum \frac{1}{n^{1.33\ldots}}$ converges
p-series $\quad p > 1$

Our series converges by limit comp. w/ $\displaystyle\sum \frac{1}{n^{1.33}}$

Eg: Does $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converge?

$\boxed{\text{Set}}$

$$a_k = \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} = \frac{k\cdot k^2\left(2-\frac{1}{k}\right)\left(1-\frac{1}{k^2}\right)}{k\cdot(k^2)^2\left(1+\frac{1}{k}\right)\left(1+\frac{4}{k^2}\right)^2}$$

$$b_k = \frac{k\cdot k^2}{k\cdot(k^2)^2} = \frac{1}{k^2}$$

(7)

compare: $\displaystyle\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\left(2-\frac{1}{k}\right)\left(1-\frac{1}{k^2}\right)}{\left(1+\frac{1}{k}\right)\left(1+\frac{4}{k^2}\right)^2}$

$$= \boxed{2 \neq 0}$$

check:
$\sum b_k = \sum \frac{1}{k^2}$ converges
p-series p 71

conclude our series converges
by limit comparison to $\sum \frac{1}{k^2}$

Limit Comparison doesn't always work

Eg: Does $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{17 n^3}$ converge?

Cannot limit compare w/ $\frac{1}{n^3}$

(check: $\lim_{n\to\infty} \frac{a_n}{b_n}$ diverges)

(5)

Direct comparison

$$\frac{2+\sin(n)}{17\cdot n^3} \leq \frac{2+1}{17\cdot n^3} = \frac{3}{17}\cdot\frac{1}{n^3}$$

AND $\sum_{n=1}^{\infty} \frac{3}{17}\cdot\frac{1}{n^3} = \frac{3}{17}\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges

p-series, $p>1$

$\Rightarrow$ ~~The~~ Our series converges by DIRECT Comparison with $\frac{3}{17}\cdot\frac{1}{n^3}$

(10)

Some limit comparisons aren't obvious

Eg: Does $\sum_{n=1}^{\infty} \sin(e^{-n})$ converge?

set: $b_n = e^{-n}$

compare: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(e^{-n})}{e^{-n}}$

$\overset{H}{=} \lim_{n\to\infty} \frac{\cos(e^{-n}) \cdot \cancel{(-e^{-n})}}{\cancel{(-e^{-n})}}$

$= \lim_{n\to\infty} \cos(e^{-n}) = 1 \neq 0$

Check $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

converges: geometric with $|r| < 1$

$\Rightarrow$ our series converges by limit comparison to $\sum_1^{\infty} e^{-n}$

clever trick: compare trig(fraction) to fraction

$\boxed{\text{must}}$ check $\lim_{n\to\infty} \frac{a_n}{b_n}$ to make sure this works.

Alternating series I

but first...

(5)

Recap of Convergence Tests

1. **Test for Divergence**

   Does the n^{th} term NOT go to 0?

   - then the SERIES diverges

2. **Special Types of Series**

   - geometric
   - p-series
   - telescoping

ALL terms ≥ 0

3. **Tests for Positive Series**

   - Integral Test.
   - Direct Comparison.
   - Limit Comparison.

Today: a special type of "sometimes negative" series

Recall: We use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{diverges}}$

Question: Does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge?

Careful: The integral test $\boxed{\text{doesn't}}$ apply ...

(5) ... some terms are negative.

In fact: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\boxed{\text{DOES converge}}$

We need a new test to show this:

(the Alternating Series Test)

(10)

Alternating Series Test

$\boxed{\text{If}}$ $a_n = (-1)^{n-1} \cdot b_n$

or $a_n = (-1)^n \cdot b_n$

b_n = positive part of a_n

AND

a_n alternates signs

And if $b_{n+1} \le b_n$

And if $\lim_{n\to\infty} b_n = 0$

Positive part b_n decreas[es] to $\boxed{0}$

$\boxed{\text{Then}}$ $\sum a_n$ converges

Notation: b_n is positive part of a_n

And b is an upside down p

(5)

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges

$a_n = \dfrac{(-1)^{n-1}}{n}$, $b_n = \dfrac{1}{n}$

Check

① a_n has alternating sign ✓

② $b_n = \dfrac{1}{n} \geq \dfrac{1}{n+1} = b_{n+1}$ ✓

③ $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the alternating series test

Eg: Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3n}{4n-1}$ converge

try alternating series test

① a_n alternates signs ☑

② $b_n = \dfrac{3n}{4n-1} \Rightarrow$ check $f'(x)$ where $f(x) = \dfrac{3x}{4x-1}$

checking derivative shows b_n is decreasing ☑

③ $\boxed{\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0}$ ✗

positive part doesn't go to 0!

$\Rightarrow$ Cannot apply alternating series test

is $\boxed{\text{NOT}}$ actually a necessary step

(10)

IN FACT we should have tried the divergence test First!

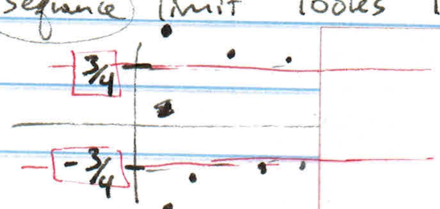
alternates sign forever

goes to $\frac{3}{4}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \boxed{(-1)^{n-1}}\, \boxed{\frac{3n}{4n-1}}$$

$\boxed{\text{DIVERGES}}$

this sequence limit looks like



$\Rightarrow$ the series $\displaystyle\sum \frac{(-1)^n\, 3n}{4n-1}$ Diverges by the $\boxed{\text{DIVERGENCE}}$ test

Eg: Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ converge?

$$a_n = (-1)^{n+1} \frac{n^2}{n^3+1}$$

$$b_n = \frac{n^2}{n^3+1}$$ the positive part of a_n

① a_n ~~is~~ alternates signs $\boxed{\checkmark}$

② b_n is decreasing

if $f(x) = \frac{x^2}{x^3+1}$

then $f'(x) =$ quotient rule $= \cdots < 0$

$\Rightarrow$ $b_n = f(n) > f(n+1) = b_{n+1}$

$\boxed{\checkmark}$

③ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 0$

$\boxed{\checkmark}$

(5)

$\Rightarrow$ the series converges

by the alternating series test

There is a nice picture behind the alternating Series test

Setup: Consider $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

look at first 3 partial sums

$S_1 = b_1$

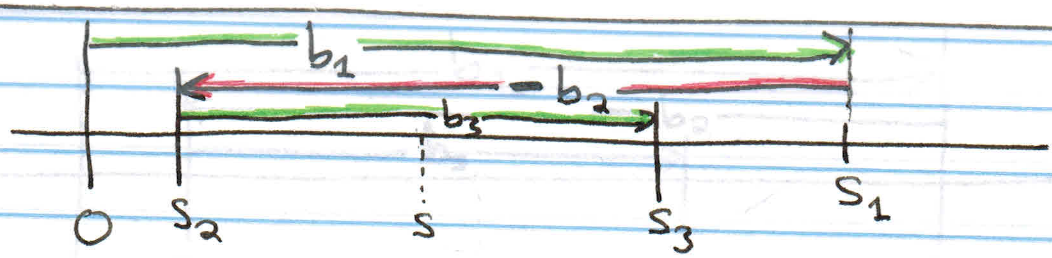
$S_2 = b_1 - b_2$

$S_3 = b_1 - b_2 + b_3$

...

Because $b_n \geq b_{n+1}$, we get the picture

picture:



Because $\lim_{n\to\infty} b_n = 0$,

the partial sums converge to a # S

3 step argument:

① even partial sums are increasing & bounded ⇒ converge

② odd partial sums are decreasing & bounded ⇒ converge

③ distance between odd & even terms → 0
the odd & even sums converge to same *

Alternating Series Test and Error Estimates

(5)

Alternating Series Test

The series $\sum a_n$ converges

IF ① $a_n = \underbrace{(-1)^{n-1}}_{\text{alternating}} \cdot \underbrace{b_n}_{\text{positive part}}$

($a_n = (-1)^n \cdot b_n$ also works)

AND ② $b_n \geq b_{n+1}$

③ $\lim_{n\to\infty} b_n = 0$

} positive part decreases to 0.

If you notice $\lim_{n\to\infty} b_n \neq 0$

Apply divergence test instead.

Sadly: we have no good "name"

for $s = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot b_n$

(5) "Theorem:" There are more #'s defined by
Series than there are "names" for #'s.

Happily: the partial sums S_n
give a nice approximation to the sum S

Even Better: For alternating series,
we can bound $\begin{pmatrix}\text{error of}\\ \text{first } n \text{ terms}\end{pmatrix} = S - S_n$

Alternating Error Bound:

Remember:

$s_1 = b_1$ ← (sum of first term)

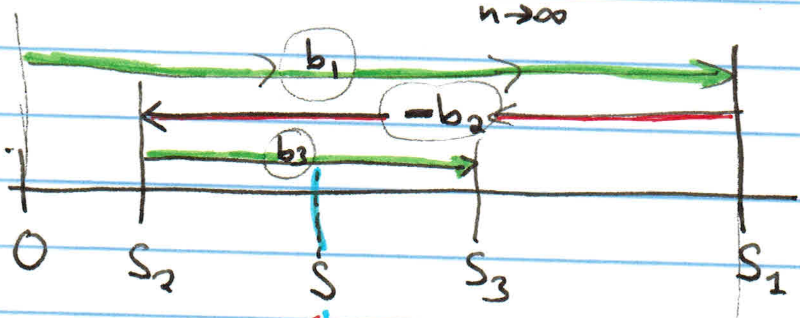
$s_2 = b_1 + (-b_2)$ ← (sum of first 2 terms)

$s_3 = b_1 + (-b_2) + b_3$ ← (sum of first 3 terms)

$\vdots$

Because $b_n \geq b_{n+1}$ and $\lim\limits_{n \to \infty} b_n = 0$

we get



partial sums converge to s

(10)

NOTICE: $|s - s_1| \leq b_2$ because "$-b_2$" overshoots s

$|s - s_2| \leq b_3$ because "$+b_3$" overshoots s

$\vdots$

$|s - s_n| \leq b_{n+1}$

This Gives us the Alternating Series Error Estimation theorem

If $s = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot b_n$ is a convergent alternating series

Then $\left| \begin{matrix} \text{error of} \\ n^{th} \text{ partial sum} \end{matrix} \right| = |s - s_n| \leq b_{n+1}$

(15)

<u>Eg:</u> [To compute $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{10^n} = S$ correctly to 3 decimal places,]

[Compute longer and longer partial sums until adding / subtracting b_{n+1} doesn't change the first 3 decimal places of S_n]

$a_n = \frac{(-1)^{n-1}}{10^n}$

$b_n = \frac{1}{10^n}$

$S_1 = \frac{1}{10} = \underset{}{\overset{b_1}{.1}}$

$S_2 = \frac{1}{10} - \frac{1}{100} = .1 - \overset{b_2}{.01} = .09$

$S_3 = \frac{1}{10} - \frac{1}{100} + \frac{1}{1{,}000} = .09 + \overset{b_3}{.001} = .091$

[redacted]

Sum of first 4 terms → $S_4 = S_3 - \frac{1}{10{,}000} = .091 - \overset{b_4}{.0001} = \boxed{.090}9$

Sum of first 5 terms → $S_5 = S_4 + \frac{1}{100{,}000} = .0909 + \overset{b_5}{.00001} = \boxed{.090}91$

<u>NOTICE:</u> Adding/subtracting $b_5 = .00001$ does <u>NOT</u> change 3rd decimal place

$\Rightarrow$ first 3 digits of sum S are $.090$ by alternating error estimate

Eg: What is the least # of terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ needed to get $\left|\begin{matrix}\text{error}\\ \text{of } S_n\end{matrix}\right| \leq .001$?

Need 2 ideas

(15) ① Notice $.001 = \frac{1}{1000} = \frac{1}{10^3}$

② Remember $\left|\begin{matrix}\text{error of}\\ S_n\end{matrix}\right| \leq b_{n+1} = \frac{1}{(n+1)^3}$

$\boxed{\text{Put together}}$

WANT $\quad b_{n+1} = \frac{1}{(n+1)^3} \leq \frac{1}{10^3}$

flip fractions ⇒ flip ≤

so we want $\quad (n+1)^3 \geq 10^3$

so we WANT $\quad n \geq 9$

[redacted] Conclude: S_9 is the first partial sum with $|\text{error}| \leq$ [redacted] $.001$

Eg: for the same series,
What is the least # of terms
to get $|\text{error}| \leq .000\,000\,001$?

i.e. $\left|\begin{matrix}\text{error}\\ \text{of } S_n\end{matrix}\right| \leq \frac{1}{10^9}$

WANT: $\left|\begin{matrix}\text{error of}\\ S_n\end{matrix}\right| \leq b_{n+1} = \frac{1}{(n+1)^9} \leq \frac{1}{(10^3)^3} \Rightarrow$ $n+1 \geq 1000$, $n \geq 999$

Absolute Convergence and the Ratio Test:

When deciding if a series converges, Think about

① Test for Divergence

② Special Series

(5)

- geometric
- p-series
- $\boxed{\text{Alternating}}$ ← did this the last 2 classes.
- telescoping

③ Tests for $\boxed{\text{positive}}$ series

- Integral test
- (DIRECT) Comparison
- Limit Comparison

(We need more tools for testing convergence of sometimes negative ~~[illegible]~~ non-alternating series!)

Today: tools for non-alternating sometimes-negative series

Eg: what does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ do?

NOTICE ① $\sin(n)$ is sometimes negative so can't apply comparison tests

(10)

② $\sin(n)$ is **NOT** alternating in sign so can't apply alternating series test

Easier sub-question:
What does $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ do?

Notice: $0 \le |\sin(n)| \le 1$

so

compare $\displaystyle\frac{|\sin(n)|}{n^2} \le \frac{1}{n^2}$

and

check $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p > 1$)

$\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges by direct comparison with $\displaystyle\sum \frac{1}{n^2}$

Theorem:
If $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}$ converges,
Then $\displaystyle\sum \frac{\sin(n)}{n^2}$ converges

$\displaystyle\sum \frac{\sin(n)}{n^2}$ converges

More Generally

Define: $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

if the series of absolute values

$\sum_{n=1}^{\infty} |a_n|$ is convergent

(10) **Theorem:**

If $\sum a_n$ converges absolutely

Then $\sum a_n$ converges.

IE: $\left[\sum |a_n| \text{ converges}\right] \Rightarrow \left[\sum a_n \text{ converges}\right]$

Eg: $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent

because $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges.

Eg: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent

because $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ converges.

Not all series are this nice!

Define: $\sum a_n$ is conditionally convergent

if $\sum a_n$ converges
and $\sum |a_n|$ diverges

(5) Eg: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent

① the series converges by the alternating series test

② $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p \le 1$).

The Integral and Comparison tests CANNOT show that a "sometimes negative" series diverges

Our last two convergence tests CAN!

Ratio Test: Let $\sum_{n=1}^{\infty} a_n$ be ANY series (such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists)

There are 3 cases:

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

Then $\sum a_n$ is absolutely convergent

(2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

Then $\sum a_n$ diverges

(doesn't even converge conditionally)

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

then we learn NOTHING

[it could do any of the 3 options
- absolutely converge
- conditionally converge
- diverge]

Eg: What does $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ do?

$$|a_n| = \left|\frac{(-2)^n}{n!}\right| = \frac{2^n}{n!}$$

① simplify the ratio!

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

(collect related terms) and (cancel)

$$= \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$$

$$= 2 \cdot \frac{\cancel{n} \cdot \cancel{(n-1)} \cdots \cancel{2} \cdot 1}{(n+1) \cdot \cancel{n} \cdot \cancel{(n-1)} \cdots \cancel{2} \cdot 1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}$$

② compute the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

$\Rightarrow$ the original series **converges absolutely** by the ratio test

Eg: What does $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{5^n}$ do?

Important!

$$|a_n| = \left|(-1)^n \frac{n^2}{5^n}\right| = \frac{n^2}{5^n}$$

simplify the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^2}{5^{n+1}}}{\frac{n^2}{5^n}} = \frac{(n+1)^2}{5^{n+1}} \cdot \frac{5^n}{n^2}$$

$$= \frac{(n+1)^2}{n^2} \cdot \frac{5^n}{5^{n+1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n^2} \cdot \frac{1}{5}$$

take the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{(n+1)^2}{n^2} \cdot \frac{1}{5}\right)$$

$$\left(\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \frac{(1+\frac{1}{n})^2}{1} = 1\right)$$

$$= (1)\left(\frac{1}{5}\right) < 1$$

$\Rightarrow$ the series converges absolutely by the ratio test

Extra Handout:

A Method for Testing

Absolute convergence / Conditional convergence / divergence

(1) Check: Does the n^{th} term of the sum go to 0 as $n \to \infty$?

If Not: the series diverges by the divergence test and you are done

(2) Check: does $\sum |a_n|$ converge?

→ you have many tools to check this

→ Run down the "series convergence test" list

If yes: the series converges absolutely & normally and you are done

(3) If $a_n \to 0$ and $\sum |a_n|$ diverges, Check: if $\sum a_n$ converges.

Notice: when this happens The series is often an alternating series.

Absolute Convergence

and the ratio & root tests

(Remember)

Absolute convergence: $\sum |a_n|$ converges
(thus $\sum a_n$ also converges)

Conditional convergence: $\sum a_n$ converges
BUT $\sum |a_n|$ diverges

Diverges: Both $\sum a_n$ **and** $\sum |a_n|$ diverge

(10)

Ratio Test:	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1 \Rightarrow \sum \lvert a_n \rvert$ converges
	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1 \Rightarrow \sum a_n$ diverges
	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1 \Rightarrow$ we learn NOTHING

(The Ratio test is also useful for positive series that are a mix of $n!$, polynomials, & exponentials)

(5)

Eg: What does $\sum_{n=1}^{\infty} n \cdot 5^{n-1} \cdot 2^{1-2n}$ do?

$$|a_n| = n \cdot 5^{n-1} \cdot 2^{1-2n} = \frac{n \cdot 5^{n-1}}{2^{2n-1}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\, 5^{(n+1)-1}}{2^{2(n+1)-1}} \cdot \frac{2^{2n-1}}{n \cdot 5^{n-1}}$$

$$= \frac{(n+1)}{n} \cdot \frac{2^{2n-1}}{2^{2n+1}} \cdot \frac{5^n}{5^{n-1}}$$

$$= \frac{(n+1)}{n} \cdot \frac{1}{2^2} \cdot 5$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{5}{4} = \frac{5}{4} > 1$$

$\Rightarrow$ the series diverges by the ratio test

If n^n appears, you will use $\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{b\cdot n}=e^{a\cdot b}}$

Eg: What does $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ do?

In Class

EITHER

Do $\sum \frac{n^n}{n!}$

OR

$\sum \frac{n!}{n^n}$

(10)

$$|a_n| = \frac{\boxed{n}^{\boxed{n}}}{\boxed{n}!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(\boxed{n+1})^{\boxed{n+1}}}{(\boxed{n+1})!}\cdot\frac{n!}{n^n}$$

$$= \frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)!}$$

$$= \frac{(n+1)^n\cdot(n+1)}{n^n}\cdot\frac{\cancel{n}\cdot\cancel{(n-1)}\cdots\cancel{2}\cdot 1}{(n+1)\,\cancel{n}\cdot\cancel{(n-1)}\cdots\cancel{2}\cdot 1}$$

$$= \left(\frac{n+1}{n}\right)^n\cdot\cancel{\frac{(n+1)}{(n+1)}} = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

a useful formula to remember!

$\Rightarrow$ the series diverges by the ratio test

If n^n appears, you will use: $\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{a\cdot b}}$

Eg: What does $\sum_{n=1}^{\infty}$ [redacted] $\frac{n!}{n^n}$ do ?

(15)

$$|a_n| = \frac{n!}{n^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!}$$

$$= \frac{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{2}\cdot 1}{\cancel{n}\cdot\cancel{(n-1)}\cdots\cancel{2}\cdot 1} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{\cancel{(n+1)}\,n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n\cancel{(n+1)}}} = \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Remember

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

compute $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim e^{n\cdot\ln(1+\frac{1}{n})} = \dots$

$\Rightarrow$ the series converges absolutely by the ratio test.

We close with one last test which is similar to the ratio test.

The Root Test

suppose $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(|a_n|\right)^{\frac{1}{n}}$ exists

(1) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$

(5) Then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely

(2) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$

Then $\sum a_n$ diverges

(3) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Then we learn <u>NOTHING</u>

<u>use</u> if "n" is in EVERY exponent

Eg: What does $\sum_{n=1}^{\infty} \frac{(n^2+1)^{5n}}{(n!)^{2n}}$ do?

Root Test

$$|a_n| = \frac{(n^2+1)^{5n}}{(n!)^{2n}} = \left(\frac{(n^2+1)^5}{(n!)^2}\right)^n$$

n is in the exponent of all terms!

so

(10)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left(\frac{(n^2+1)^5}{(n!)^2}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{(n^2+1)^5}{(n!)^2}$$

(factor out fastest on top & bottom)

$$= \lim_{n\to\infty} \frac{(n^2)^5\left(1+\frac{1}{n^2}\right)^5}{(n!)^2 \cdot 1}$$

Remember: $n! \gg n^{10} = (n^2)^5$

AND clearly $(n!)^2 \gg n!$

so $= 0 < 1$

$\Rightarrow$ the series converges absolutely

Extra Ratio Test Example (if time)

Eg: What does

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!} \quad \text{do ?}$$

Ratio Test:

$$|a_n| = \frac{n^{2n}}{(2n)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{2(n+1)}}{\left(2(n+1)\right)!} \cdot \frac{(2n)!}{n^{2n}}$$

$$= \frac{(2n)!}{(2n+2)!} \cdot \frac{(n+1)^{2n+2}}{n^{2n}}$$

$$= \frac{\cancel{(2n)}\cancel{(2n-1)}\cdots\cancel{2}\cdot 1}{(2n+2)(2n+1)\cancel{(2n)}\cancel{(2n-1)}\cdots\cancel{2}\cdot 1} \cdot \left(\left(\frac{n+1}{n}\right)^n\right)^2 \cdot (n+1)^2$$

$$= \frac{1}{(2n+2)(2n+1)} \cdot \left(\left(1+\frac{1}{n}\right)^n\right)^2 \cdot (n+1)^2$$

$$\lim_{n\to\infty} \underbrace{\frac{(n+1)(n+1)}{(2n+2)(2n+1)}}_{\frac{1}{4} \text{ by L'Hopital's twice}} \cdot \Bigg(\underbrace{\left(1+\frac{1}{n}\right)^n}_{e \text{ by limit formula}}\Bigg)^2 = \frac{e^2}{4}$$

and $\quad e > 2 \Rightarrow \dfrac{e^2}{4} > 1$

$\Rightarrow$ the series diverges

Summary of Series Convergence Tests

(1) Test for divergence

If n^{th} term doesn't go to 0
Then the sum diverges

(2) Special Types of series

- geometric
- p-series
- alternating series
- telescoping series

(10)

(3) Tests for positive series

- Integral test

use if you think
"I could integrate that!"

- (Direct) comparison
- Limit comparison

squint and ask
"what really matters here?"

(4) Absolute Tests

- ratio test

for mixes of $n!$, polynomials, & exponentials

- root test

If ~~the~~ you can factor out $(\text{—})^n$

(Absolute / conditional / divergence :)

Series with negative terms
can do strange things

→ Ratio & root tests speak with finality
"converges absolutely" or "diverges"

→ Integral & comparison tests
can show absolute convergence

(5)

but

cannot decide between divergence
& conditional convergence

→ other methods (eg. Alternating series test)
needed to show conditional convergence.

Eg: What does $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \cdot \ln(n)}$ do?

[redacted]

notice: alternating & positive part decreases to 0

$\Rightarrow$ the series **converges** by alternating series test

Qn: does this converge absolutely or conditionally?

think: I could integrate $|a_n| =$ [redacted] $\dfrac{1}{n \cdot \ln(n)}$

Integral test:

(10)

$$\int_2^{\infty} \frac{1}{x \cdot \ln(x)}\, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \cdot \ln(x)}\, dx$$

$$\left(\begin{array}{l} u = \ln(x) \\ du = \frac{1}{x}\, dx \end{array} \right)$$

careful here. see why in 1 sec

$$= \lim_{t \to \infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\, du$$

$$= \lim_{t \to \infty} \Big[\ln(u) \Big]_{\ln(2)}^{\ln(t)}$$

2 ln's here!

$$= \lim_{t \to \infty} \Big[\ln\big(\ln(t)\big) \Big] - \ln\big(\ln(2)\big)$$

the integral Diverges to ∞

$\sum |a_n|$ [redacted] $= \sum \dfrac{1}{n \cdot \ln(n)}$ diverges by integral test

$\Rightarrow$ $\boxed{\sum a_n \text{ converges conditionally}}$

[We'll do a series convergence worksheet in class on Wed]

[Close today with some more tricky ratio/root test problems]

Eg: What does $\displaystyle\sum_{n=1}^{\infty} \frac{\left(1+\frac{2}{n}\right)^{n^2} \cdot n^{2n}}{\left(n^5+2\right)^{\frac{n}{2}}}$ do?

root test:

$$|a_n| = \frac{\left(1+\frac{2}{n}\right)^{n \cdot n} \cdot n^{2 \cdot n}}{\left(n^5+2\right)^{n \cdot \frac{1}{2}}}$$

$$= \left(\frac{\left(1+\frac{2}{n}\right)^{n} \cdot n^2}{\left(n^5+2\right)^{\frac{1}{2}}}\right)^{n}$$

(10)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{\left(1+\frac{2}{n}\right)^{n} \cdot n^2}{\left(n^5+2\right)^{\frac{1}{2}}}\right)^{\frac{n}{n}}$$

$$= \lim_{n\to\infty} \frac{\overbrace{\left(1+\frac{2}{n}\right)^{n}}^{\text{not going to } \infty} \cdot n^2}{\left(n^5+2\right)^{\frac{1}{2}}}$$

Pull out fastest growing terms

$$= \lim_{n\to\infty} \frac{n^2}{n^{\frac{5}{2}}} \cdot \frac{\left(1+\frac{2}{n}\right)^{n}}{\left(1+\frac{2}{n^5}\right)^{\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{n^{.5}}}_{\to 0} \cdot \frac{\boxed{\left(1+\frac{2}{n}\right)^{n}} \to e^2}{\underbrace{\left(1+\underbrace{\frac{2}{n^5}}_{\to 0}\right)^{\frac{1}{2}}}_{\to 1}}$$

$\to e^2$ by formula from last time

$$= 0 < 1$$

$\Rightarrow$ The series converges (absolutely) by root test.

Eg: What does $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{2^n\,(n!)^2}$ do?

ratio test

(10)

$$|a_n| = \frac{(2n+1)!}{2^n\,(n!)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(2\,(n+1)+1\right)!}{2^{(n+1)}\left((n+1)!\right)^2} \cdot \frac{2^n\,(n!)^2}{(2n+1)!}$$

$$= \frac{(2n+3)!}{(2n+1)!} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n!\;n!}{(n+1)!\cdot(n+1)!}$$

$$\boxed{\begin{aligned}(2n+3)! &= (2n+3)(2n+2)(2n+1)(2n)\cdots 2\cdot 1\\ (2n+1)! &= \cancel{(2n+3)(2n+2)}\cdot(2n+1)(2n)\cdots 2\cdot 1\end{aligned}}$$

$$= \frac{(2n+3)(2n+2)}{1} \cdot \frac{1}{2^1} \cdot \frac{1}{(n+1)} \cdot \frac{1}{(n+1)}$$

$$= \frac{(2n+3)(2n+2)}{2\cdot(n+1)\cdot(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2n+3)(2n+2)}{2(n+1)(n+1)} = \lim_{n\to\infty}\frac{4n^2(1+\frac{3}{2n})(1+\ldots}{2n^2(1+\frac{1}{n})(1+\ldots}$$

$$= 2 > 1$$

$\Rightarrow$ the series diverges by the ratio test

Sequence Limits

Notation: $\lim_{n\to\infty} a_n$, limit of $\{a_n\}$.

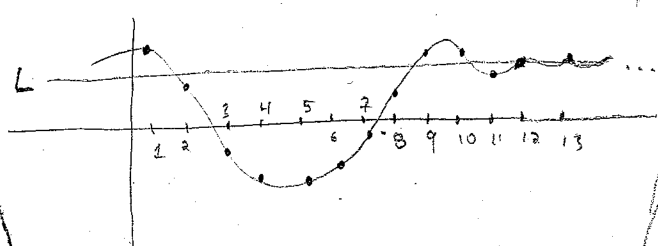
NOTE: n is ALWAYS a counting #

Sequence Limits that are like function limits

If $a_n = f(n)$

and $\lim_{x\to\infty} f(x) = L$

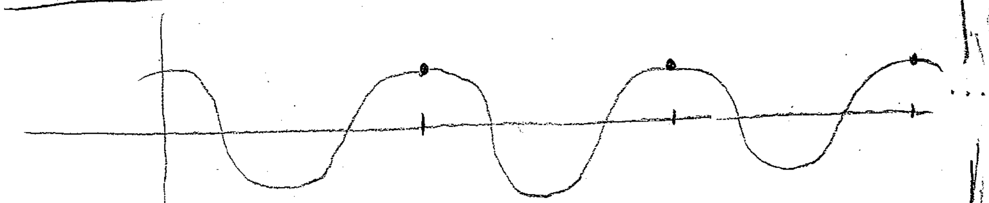
Then $\lim_{n\to\infty} a_n = L$



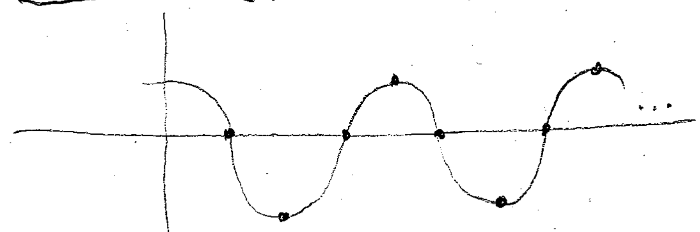
NO overlap!

Sequence limits that are NOT like function limits

Some converge: $a_n = \cos(2\pi n)$



Some diverge: $a_n = \cos\left(\frac{\pi n}{2}\right)$



Series

Notation: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$

Series Convergence tests

1. Divergence test
2. Special Series
   - geometric
   - p-series
   - alternating
   - telescoping
3. tests for positive series
   - Direct comparison
   - limit comparison
   - Integral test
4. Ratio & Root tests

Absolute convergence/ conditional convergence

Basic Series Methods

→ Geometric sums

→ Series arithmetic

→ alternating error estimation

→ Partial Sums

$S_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$

the whole series is the limit of its partial sums

$\sum_{n=1}^{\infty} a_i = \lim_{n\to\infty} S_n$

(these ideas and methods overlap)

KEY:

Involves a sequence limit

Involves a function limit

Function Limits

<u>Notation:</u> x, y, z can be any number

Applying function limit tricks to certain sequence limits

Applying function limit tricks

— to certain sequence limits

Tricky Types of Limits

→ Indeterminate of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

(⇒ use L'Hopital's Rule or clever rewriting)

→ Indeterminate of type $0 \cdot \infty$, 1^{∞}, ∞^{0}, $\infty - \infty$

(⇒ rewrite until you can use L'Hopital's Rule)

→ Complex combinations of functions

⇒ think carefully.

⇒ use rates of growth to find the fastest term?

⇒ rewrite to be easier to understand?

Improper Integrals

1. turn into a limit of proper integrals
2. integrate
3. take limit