Work out 1 problem from exam (16)

6.8: Under-determined Limits
(indeterminite forms)

(As we work with limits of functions and sequences we will often encounter ∞.)

(In Calculus,)
∞ is not a number,
it is a limit (bigger & bigger).

So, you cannot do arithmetic with ∞.

Some "computations" work:

(7.)

$\lim_{x\to\infty} (x+e^x) = \infty$	$\Rightarrow$	$\infty + \infty = \infty$
$\lim_{x\to\infty} (x\cdot e^x) = \infty$	$\Rightarrow$	$\infty \cdot \infty = \infty$
$\lim_{x\to\infty} (x-(-e^x)) = \infty$	$\Rightarrow$	$\infty - (-\infty) = \infty$

but most do not!

If a limit produces $\frac{\infty}{\infty}$,
$\frac{0}{0}$,
$\infty - \infty$, etc.
the limit is indeterminite

(the limit is under-determined,
and must be found via other methods)

Eg: Compare $\lim_{x\to\infty} \frac{x}{x^2}$, $\lim_{x\to\infty} \frac{x^3}{x}$, $\lim_{x\to\infty} \frac{x}{x}$.

Two Most common cases: (of indeterminate forms)

(I) $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ when $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$

(Indeterminate of type $\frac{0}{0}$)

and

(II) $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ when $\lim\limits_{x\to a} f(x) = \pm\infty$

and $\lim\limits_{x\to a} g(x) = \pm\infty$

(Indeterminate of type $\frac{\infty}{\infty}$)

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L'Hopital's Rule

A major tool

IF $g'(x) \neq 0$ near a (it is possible that $g'(a) = 0$ AT a)

And IF $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is in case (I) or (II)

(if $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

THEN $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

Examples of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$:

$$\lim_{x\to 1} \frac{\ln(x) \to 0}{x-1 \to 0} \overset{H}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$$

pay attention to the limit

must check before L'Hopital's rule

I will write "H" over =′s obtained by L'Hopital's.

Reason: Need algebra AND L'Hopital to solve these

the "$\overset{H}{=}$" identifies the calc method

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$$\lim_{x\to\infty} \frac{e^x \to \infty}{x^2 \to \infty} \overset{H}{=} \lim_{x\to\infty} \frac{e^x \to \infty}{2x \to \infty} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$$\lim_{x\to\infty} \frac{\ln(x) \to \infty}{\sqrt[3]{x} \to \infty} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$

$x^{1/3}$

Rewrite using ALGEBRA, (not L'Hopital's)

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}}$$

Indeterminate Product:

have type $0 \cdot (\pm\infty)$

(function getting very small) (function getting very big)

to solve, **rewrite** $f \cdot g$ **as** $\dfrac{f}{\frac{1}{g}}$ or $\dfrac{g}{\frac{1}{f}}$

5

(Be careful to make the correct quotient long)

Eg: $\lim\limits_{\boxed{x \to 0^+}} \overset{\nearrow 0}{x} \cdot \overset{\nearrow -\infty}{\ln(x)} = \lim\limits_{x \to 0^+} \dfrac{\ln(x) \nearrow -\infty}{\frac{1}{x} \searrow \infty}$

Algebra

$$\overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} (-x) = 0$$

Algebra tricks can save time:

Eg:

$$\lim_{x\to\infty} \frac{x^7+1}{5x^7+x+2} = \lim_{x\to\infty} \frac{x^7\left(1+\frac{1}{x^7}\right)}{x^7\left(5+\frac{1}{x^6}+\frac{2}{x^7}\right)}$$

NOTE: NO L'Hopital

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x^7}}{5+\frac{1}{x^6}+\frac{2}{x^7}}$$

$\frac{1}{x^7} \to 0$, $\frac{1}{x^6} \to 0$, $\frac{2}{x^7} \to 0$

$$= \frac{1}{5}$$

Eg:

$$\lim_{x\to\infty} \frac{2e^{x}-e^{-x}}{3e^{2x}+1}$$

$$= \lim_{x\to\infty} \frac{e^{x}\left(2-e^{-2x}\right)}{e^{2x}\left(3-e^{-2x}\right)} = 0$$

$\frac{1}{e^x} \to 0$, $\frac{2-e^{-2x}}{3-e^{-2x}} \to \frac{2}{3}$

$$= \lim_{x\to\infty} \underbrace{\frac{1}{e^x}}_{\to 0} \cdot \underbrace{\left(\frac{2-e^{-2x}}{3-e^{-2x}}\right)}_{\to 2/3}$$

$e^{-2x} \to 0$

$$= 0$$

NOTE: NO L'Hopital.

Two More tricky Examples:

Eg: $$\lim_{x\to 0} \frac{\tan(x) - x \to 0}{x^3 \to 0}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\sec^2 x - 1 \to 0}{3x^2 \to 0}$$

10 $$\overset{H}{=} \lim_{x\to 0} \frac{2\cdot\sec(x)\cdot\sec(x)\tan x}{6x}$$

hard to tell what is happening

collect related terms

$$= \lim_{x\to 0} \left(\underbrace{\frac{2}{6}\cdot \overbrace{\sec^2(x)}^{\to 1}}_{\to \frac{2}{6} \neq 0} \cdot \frac{\tan(x)}{x} \right)$$

[from calc 1: Can pull out NONZERO ~~[illegible]~~ FINITE limits]

$$= \frac{2}{6}\cdot \lim_{x\to 0} \frac{\tan(x) \to 0}{x \to 0}$$

$$\overset{H}{=} \frac{2}{6}\cdot \lim_{x\to 0} \frac{\sec^2(x) \to 1}{1}$$

$$= \frac{2}{6}\cdot 1 = \frac{1}{3}$$

Eg: $$\lim_{x\to \pi^-} \frac{\sin(x) \to 0}{1-\cos(x) \to 1-(-1) = 2} = 0$$

(pay attention to the limit)

[Cannot Apply L'Hopital's here. Trying would incorrectly give $-\infty$]

Indeterminate Differences

If $\lim\limits_{x\to a} f(x) = \infty = \lim\limits_{x\to a} g(x)$

Then $\lim\limits_{x\to a} \left[f(x) - g(x)\right]$

is indeterminate of type $\infty - \infty$

why not determined?

$$\lim_{x\to\infty} (x^2 - x) = \infty$$

$$\lim_{x\to\infty} (x - x^2) = -\infty$$

$$\lim_{x\to\infty} (x - x) = 0$$

---

compute by turning into a fraction of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Eg: $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} \overset{\nearrow\infty}{\sec(x)} - \overset{\nearrow\infty}{\tan(x)}$

how to get fraction from $\sec(x)$ & $\tan(x)$?

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right)$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(\frac{\overset{\to 0}{1-\sin(x)}}{\underset{\to 0}{\cos(x)}}\right)$$

$$\overset{H}{=} \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\overset{\to 0}{-\cos(x)}}{\underset{\to 1}{-\sin(x)}} = 0$$

Indeterminate Powers

- $\lim_{x \to a} \left(f(x)\right)^{g(x)}$ is indeterminate

  IF it is of type 0^0, ∞^0, or 1^∞

  (goes to 0 or 1?) (to ∞ or 1?) (to 1 or ∞?)

Two Methods for Computing

Ⓐ ① let $y = \left(f(x)\right)^{g(x)}$

② compute $\lim_{x \to a} \ln(y) = \left(\lim_{x \to a} g \cdot \ln(f) = \right) A$

③ Answer $\lim_{x \to a} y = \lim_{x \to a} e^{\ln(y)} = e^A$

Ⓑ ① Recall $\left(f(x)\right)^{g(x)} = e^{g(x) \cdot \ln(f(x))}$

② compute $\lim_{x \to a} g(x) \cdot \ln\left(f(x)\right) = A$

③ Answer $\lim_{x \to A} \left(f(x)\right)^{g(x)} = e^A$

Eg: $\lim_{\boxed{x \to 0^+}} \Big(\underbrace{1 + \sin(4x)}_{\to 1} \Big)^{\overbrace{\cot(x)}^{\to \infty}}$ $\boxed{\text{of type } 1^{\infty}}$

(pay attention to direction)

$$\boxed{\begin{aligned}
&\underline{\text{set}} \quad y = \big(1 + \sin(4x)\big)^{\cot(x)} \\
&\underline{\text{then}} \quad \ln(y) = \ln\Big(\big(1+\sin(4x)\big)^{\cot(x)} \Big) \\
&\qquad\qquad\quad = \underbrace{\cot(x)}_{\left(\frac{1}{\tan(x)}\right.} \cdot \ln\big(1 + \sin(4x)\big) \\
&\underline{\text{so}} \quad \lim_{x \to 0^+} \ln(y) = \boxed{\lim_{x \to 0^+}} \frac{\overbrace{\ln(1 + \sin(4x))}^{\to 0}}{\underbrace{\tan(x)}_{\to 0}}
\end{aligned}}$$

[DON'T FORGET]

$$\overset{H}{=} \lim_{x \to 0^+} \left[\frac{\left(\frac{1}{\underbrace{1+\sin(4x)}_{\to 1}}\right) \cdot \overbrace{\cos(4x)}^{\to 1} \cdot 4}{\underbrace{\sec^2(x)}_{\to 1}} \right]$$

$$\underline{\text{so}} \quad \lim_{x \to 0^+} \ln(y) = 4$$

Not the final answer!

<u>conclude</u>

$$\lim_{x \to 0^+} \big([redacted]\ 1 + \sin(4x)\big)^{\cot(x)} = \lim_{x \to 0^+} y$$

$$= \lim_{x \to 0^+} e^{\ln(y)} = \boxed{e^4}$$

Relative Rates of Growth: ← (handout, eg's, & slns on Math 141 page)

Soon (for sequences and series) we will use the fact that

$$\lim_{x\to\infty} \frac{\text{crazy fn}}{\text{crazy fn}} \approx \lim_{x\to\infty} \frac{\text{fastest of top}}{\text{fastest of bottom}}$$

10 min

Details and Examples Next week.

Today, we define "fastest"

Suppose $f(x)$ and $g(x)$ **both**

1. are eventually positive
2. go to ∞

Then

$$\boxed{\begin{array}{l} \underline{f \text{ grows faster than } g} \quad (f \gg g) \\[1em] \iff \displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \\[1em] \iff \displaystyle\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \end{array}}$$

---

check this yourself and/or see rates of growth handout

Important to know:

$$\boxed{\underbrace{\ln(x)}_{\text{logs}} \ll \underbrace{x \ll x^2 \ll x^3}_{\text{Polynomials}} \ll \underbrace{2^x \ll e^x \ll 3^x}_{\text{exponentials}} \ll \underbrace{x^x}_{\text{crazy fast!}}}$$

Nice to know: $x \ll x \cdot \ln(x) \ll x^2$

We also say

$$\boxed{\begin{array}{l} f(x) \text{ and } g(x) \ \underline{\text{grow at the same rate}} \quad (f \simeq g) \\ \text{if } \displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = M \neq 0 \end{array}}$$

← nonzero finite #

10 min

Eg: $3x^2$ and $x^2 + 19x - 4$ grow at the same rate

$$\lim_{x\to\infty} \frac{3x^2}{x^2+19x-14} = \lim_{x\to\infty} \cancel{\frac{x^2}{x^2}} \cdot \frac{3}{\left(1 + \frac{19}{x} - \frac{14}{x^2}\right)} = 3 \neq 0$$

$\frac{19}{x} \to 0$, $\frac{14}{x^2} \to 0$

pull out fastest terms on top and bottom

Same argument:

① all degree n polynomials grow at same rate

② $(e^x + x^2)$ grows at same rate as $(2 \cdot e^x)$

Qn: Why is this the same argument?

7.8 - Improper Integrals

Remember: ∞ is not a #
it is a "limit".

6 min

What does $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ mean?

it means $\displaystyle\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$

you will lose MAJOR pts if you don't ALWAYS write this FIRST

formally:

Define (Infinite Integrals) - 3 cases

(1) if $\displaystyle\int_a^{\boxed{t}} f(x)\,dx$ exists for $\boxed{\text{each } t}$ $\geq a$

then $\displaystyle\int_a^{\boxed{\infty}} f(x)\,dx = \lim_{\boxed{t}\to\infty}\int_a^{\boxed{t}} f(x)\,dx$

(2) If $\displaystyle\int_{\boxed{t}}^{b} f(x)\,dx$ exists for $\boxed{\text{each } t}$ $\le b$

then $\displaystyle\int_{\boxed{-\infty}}^{b} f(x)\,dx = \lim_{\boxed{t}\to-\infty} \int_{t}^{b} f(x)\,dx$

9 min

(3) If $\displaystyle\int_{a}^{\infty} f(x)\,dx$ and $\displaystyle\int_{-\infty}^{a} f(x)\,dx$

are convergent (if the limit exists)

NOTE
you can pick
any # a
to use here

then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

If the limit exists and is a finite #
the integral is called convergent

If ANY of the limits do NOT exist,

the whole integral is divergent

always Step # 1

Eg: $$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t$$

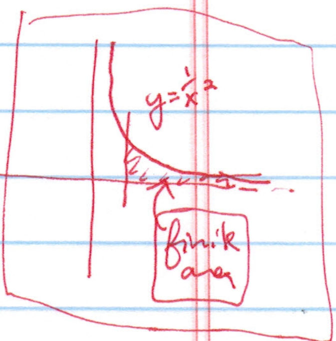
10 min

$$= \lim_{t\to\infty} \Big(\ln|t| - \ln|1|\Big) = \infty$$

the limit does not converge

$\Rightarrow$ the integral is DIVERGENT

Eg: $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$



$$= \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{t} - \frac{-1}{1}\right) = 1$$

the integral is CONVERGENT

<u>Eg:</u> $\displaystyle\int_{-\infty}^{\infty} x \, dx$

(it is helpful to pick $a = 0$)

$$= \int_{-\infty}^{0} x \, dx + \int_{0}^{\infty} x \, dx$$

diverges to $-\infty$

diverges to ∞

$\Rightarrow$ the original integral <u>DIVERGES</u>.

<5 mm

Eg: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

(again helpful to pick $a=0$)

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\left[\int_{t}^{0} \frac{1}{1+x^2}\,dx\right] + \lim_{t\to\infty}\left[\int_{0}^{t} \frac{1}{1+x^2}\,dx\right]$$

...

$$= \lim_{t\to-\infty}\Big(\underbrace{\tan^{-1}(0)}_{=0} - \underbrace{\tan^{-1}(t)}_{\to(-\pi/2)}\Big) + \lim_{t\to\infty}\Big(\underbrace{\tan^{-1}(t)}_{\to\frac{\pi}{2}} - \underbrace{\tan^{-1}(0)}_{=0}\Big)$$

$$= -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} = \pi.$$

7.8 Improper Integrals II

on Friday, looked at Infinite integrals

Eg: $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

Texbook calls these Type 1

Type 2: Inside is not continuous

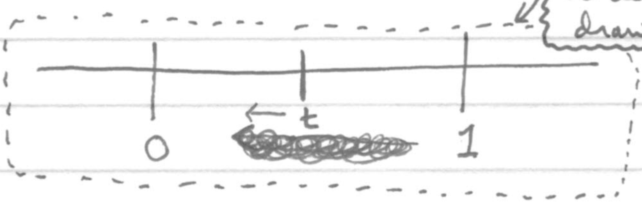
[redacted] (Discontinuous [redacted] Integrands)

$$\int_0^1 \frac{1}{x}\,dx \quad \text{is } \underline{\text{IMPROPER}}$$

because $\frac{1}{x}$ is Not continuous at 0

Define: $$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx$$

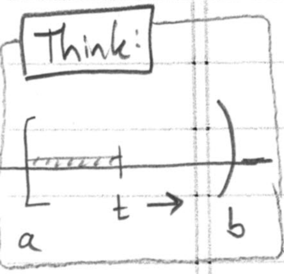
To decide direction, draw # line



Again: 3 formal Cases

(a) If f is continuous on $[a, b)$

but NOT at b,

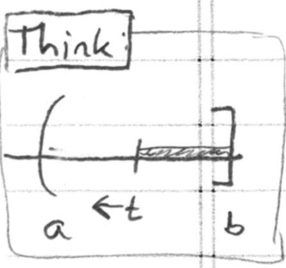


Then $$\int_a^b f(x)\,dx = \lim_{\boxed{t \to b^-}} \int_a^t f(x)\,dx$$

[redacted]

(b) If f is continuous on $(a, b]$

but NOT at a,

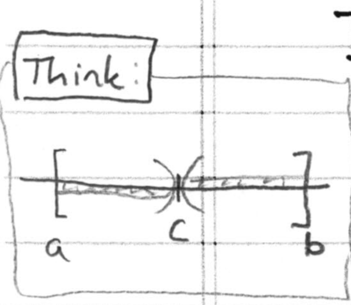


Then $$\int_a^b f(x)\,dx = \lim_{\boxed{t \to a^+}} \int_t^b f(x)\,dx$$

(c) If f is discontinuous at c

and if $a < c < b$

and if $\int_a^c f(x)\,dx$ [redacted] & $\int_c^b f(x)\,dx$ both exist as finite #'s



Then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Define: integrals are convergent
if all limits exist and are finite #'s.

otherwise integrals are divergent

Eg: $$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

[Note: improper because $\frac{1}{\sqrt{x-2}}$ is discontinuous at $x=2$
(∞)]

always first step

$$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{x\to 2^+} \left(\int_t^5 (x-2)^{-1/2}\,dx\right) = \lim_{x\to 2^+} \left[\frac{(x-2)^{1/2}}{\frac{1}{2}}\right]_t^5$$

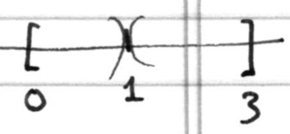
(continuous at $x=2$
so can just plug it in)

$$= 2\cdot(5-2)^{1/2} - 2\cdot(2-2)^{1/2} = 2\sqrt{3}$$

the integral is **convergent**.
(converges to $2\sqrt{3}$)

$$\underline{\text{Eg:}} \quad \int_0^3 \frac{1}{x-1}\,dx$$

case II(c)



[**WARNING:** must $\boxed{\text{see}}$ that $\frac{1}{x-1}$ is $\boxed{\text{NOT}}$ continuous at $x=1$]

(not noticing this discont would give $\ln(2)$)

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx$$

$$\underbrace{= \lim_{t\to 1^-} \Big[\underbrace{\ln|t-1|}_{\to -\infty} - \underbrace{\ln|0-1|}_{0}\Big]}_{\text{this half diverges}} + $$

$\Longrightarrow$ the <u>whole</u> integral diverges

(doesn't matter what the right side does
for integrals, discont + discont = discont)

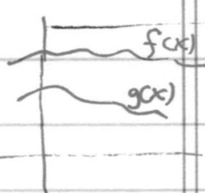
Slightly Change Gears:

Improper integrals can diverge or converge

what we have been doing

ONE WAY to decide what an integral does
is (1) turns it into a limit
(2) find anti-derivative
(3) take the limit

you can ALSO check divergence/convergence
by comparing it to an (easier) integral



Comparison test (for integrals)

If $f(x) \geq g(x) \geq 0$ for each $x \geq a$

f(x) squeezes down on g(x), making $\int g(x)dx$ finite

(1) IF $\int_a^\infty f(x)\,dx$ converges

THEN $\int_a^\infty g(x)\,dx$ converges

g(x) pushes up on f(x), making $\int f(x)dx$ infinite

(2) IF $\int_a^\infty g(x)\,dx$ diverges

THEN $\int_a^\infty f(x)\,dx$ diverges

Using the comparison test

say:

(• requires cleverness
• often straightforward comparisons
• often comparing fractions)

Eg: $$\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx$$

Notice:

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

AND remember

$$\int_1^{\infty} \frac{1}{x}\,dx \text{ diverges}$$

(check it!)

the smaller integral $\int \frac{1}{x}dx$ pushes up on $\int \frac{1+e^{-x}}{x}dx$

forcing it to diverge

Diverges by the comparison test

Useful Tricks for Fractions:

Get a <u>smaller fraction</u> by

(1) making the top smaller

(2) making the bottom larger

and use to show <u>divergence</u>.

Get a <u>larger fraction</u> by

(1) making the top bigger

(2) making the bottom smaller

and use to show <u>convergence</u>.

Some Extra Tricks

for <u>all</u> x

$$0 \leq \sin^2(x) \leq 1$$

$$0 \leq \cos^2(x) \leq 1$$

for all $\underline{x \geq 0}$

$$0 \leq e^{-x} \leq 1$$

$$0 \leq \tan^{-1}(x) \leq \pi/2$$

Eg: $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^6+1}}\,dx$

Idea 1

Compare:

compare

$$\frac{x}{\sqrt{x^6+1}} \le x$$

and note $\int_1^\infty x\,dx$ diverges

A LARGER function DIVERGES

so we learn NOTHING

Idea 2

Compare:

compare

$$\frac{x}{\sqrt{x^6+1}} \le \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$$

and remember

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges (check it!)

conclude: A larger ~~function~~ integral converges

so $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^6+1}}\,dx$ converges by comparison test

Part I

11.1 - Infinite Sequences (infinite LISTS)

(5 min)

(we now shift gears.
our goal: better understand & approximate
the fns we've used so far)

our goal:

$$e^2 = 1 + 2 + \frac{2^2}{2\cdot 1} + \frac{2^3}{3\cdot 2\cdot 1} + \frac{2^4}{4\cdot 3\cdot 2\cdot 1} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2\cdot 1} + \cdots$$

Need first:

- sequences – $\boxed{\text{list}}$ ∞-many terms

$$\left\{1, 2, \frac{2^2}{2}, \ldots\right\}$$

- Series – $\boxed{\text{add}}$ the ∞-many terms

$$1 + 2 + \frac{2^2}{2} + \cdots$$

- After exam 1 series w/ variables x^n.

(5 min)

Define: A <u>sequence</u> is an $\boxed{\text{infinite list of \#'s}}$ with a fixed order

$$\left\{ a_1,\ a_2,\ a_3,\ a_4,\ \dots,\ \boxed{a_n},\ \dots \right\}$$

first # — second # — n^{th} #

If the n^{th} term is a_n,

we often write $\{a_n\}_{n=1}^{\infty}$ for this sequence

this LIST.

Eg: $\{2, 3, 4, 5, \dots\} = \{n+1\}_{n=1}^{\infty}$

$\boxed{a_1 = 1+1}$ $\boxed{a_2 = 2+1}$ $\boxed{a_3 = 3+1}$

$a_n =$ [redacted] $n+1$ [redacted]

Eg: $\{2, 4, 6, 8, \dots\} = \{2\cdot n\}_{n=1}^{\infty}$

a_1 a_2 a_3

$a_1 = 2\cdot 1$
$a_2 = 2\cdot 2$
$a_3 = 2\cdot 3$
$\vdots$

$\Rightarrow a_n = 2\cdot n$

a list of #'s

<u>Define</u>: A sequence $\{a_n\}_{n=1}^{\infty}$

<u>converges to a limit L</u>

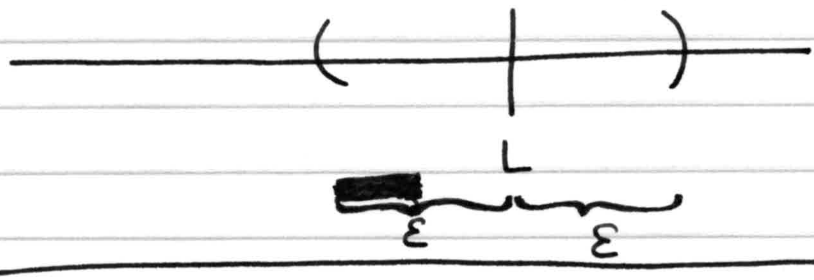
[redacted]

<u>If</u> for any challenge # $\varepsilon > 0$

there is a stage

such that all later terms a_n

(10) are in the interval



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list of #'s

<u>Eg</u>: The sequence

$$\left\{\sqrt[n]{5}\right\}_{n=1}^{\infty} = \left\{ 5^{1}, 5^{1/2}, 5^{1/3}, \ldots \right\}$$

converges to 1

<u>Eg</u>: The sequence

$$\left\{\cos(\pi n)\right\}_{n=1}^{\infty} = \left\{ 1, -1, 1, -1, \ldots \right\}$$

diverges (never settles down).

How to compute Limits?

(one type of thing)

For most sequences

a_n is defined using a function $f(x)$ (another type of thing)

Plugging in n^{th} counting number gives $a_n = f(n)$

(10)

Useful Theorem:

If a_n is defined using $f(x)$

AND if $\lim_{x\to\infty} f(x)$ exists or $= \infty$

Then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

counting #'s

all #'s

Eg: consider the sequence

$$\left\{ \frac{\ln(n)}{n} \right\}_{n=1}^{\infty} = \left\{ \frac{\ln(1)}{1}, \frac{\ln(2)}{2}, \dots \right\}$$

a_1 a_2

then $a_n = \frac{\ln(n)}{n}$

AND $\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

counting # all #'s

so $\lim a_n = \lim \frac{\ln(x)}{x} = 0$

When the limit of the function <u>diverges</u>,

must <u>think carefully</u>

(5 min)

<u>Eg:</u> the sequence

$$\left\{ \cos(2\pi n) \right\}_{n=1}^{\infty} = \{1, 1, 1, \dots\}$$

counting #'s

converges to 1

$$\boxed{\underline{\text{But}} \quad \lim_{x\to\infty} \cos(2\pi x) \text{ diverges}}$$

all #'s

<u>Notation:</u>

always use $x, y, z, \dots$ for real #'s

and use $n, i, j, \dots$ for counting #'s.

Present ~~1st~~ time
Otherwise, $\boxed{\text{will be on}}$ Wed's worksheet

(5 min)

Common Sequences:

the odd #'s

$$\{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, 7, \ldots\}$$

$$a_n = 2n-1$$

alternating sign:

$$\{(-1)^{n-1}\}_{n=1}^{\infty} = \{1, -1, 1, -1, \ldots\}$$

$$a_n = (-1)^{n-1}$$

factorial:

$$\{n!\}_{n=1}^{\infty}$$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2\cdot 1 \\ a_3 &= 3\cdot 2\cdot 1 \\ a_4 &= 4\cdot 3\cdot 2\cdot 1 \\ &\vdots \\ a_n &= n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \end{aligned}$$

$$= \{1, 2, 6, 24, \ldots\}$$

Sequences II

Today's ~~[illegible]~~ class has 3 parts

→ more on writing out sequences ← listing ~~[illegible]~~ numbers

→ more limits of sequences

→ workshet (→ try it yourself!)

(5)

Defining Sequences Using Recipes

(Not all sequences have "plug in n to get a_n" defn)

Fibonacci Sequence

the recipe
$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_{n+2} = a_n + a_{n+1} \quad \text{for } n \geq 1 \end{cases}$$

$$a_3 = a_1 + a_2 = 1 + 1 = 2$$
$$a_4 = a_2 + a_3 = 1 + 2 = 3$$
$$a_5 = a_3 + a_4 = 2 + 3 = 5$$
$$\dots$$

$$\{\underset{a_1}{1}, \underset{a_2}{1}, \boxed{\underset{a_3}{2}, \underset{a_4}{3}}, 5, \dots\}$$

This is called recursion.

(def'n of a_n DEPENDS ON previously defined values)

Sometimes Simplify First.

Skip if def'd last time

Remember: $1! = 1$

$2! = 2\cdot 1$

$3! = 3\cdot 2\cdot 1$

$4! = 4\cdot 3\cdot 2\cdot 1$

$\dots$

$n! = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

(define $0! = 1$)

(5)

$$\left\{\frac{(n-1)!}{(n+1)!}\right\}_{n=1}^{\infty} = \left\{\frac{1}{(n+1)n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \dots\right.$$

$$a_1 = \frac{0!}{2!} = \frac{1}{2\cdot 1}$$

$$a_2 = \frac{1!}{3!} = \frac{1}{3\cdot 2\cdot 1}$$

$$a_3 = \frac{2!}{4!} = \frac{\cancel{2\cdot 1}}{4\cdot 3\cdot \cancel{2\cdot 1}} = \frac{1}{4\cdot 3}$$

$$a_4 = \frac{3!}{5!} = \frac{\cancel{3\cdot 2\cdot 1}}{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}} = \frac{1}{5\cdot 4}$$

$\dots$

$$a_n = \frac{\cancel{(n-1)(n-2)}\cdots\cancel{2\cdot 1}}{(n+1)(n)\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{2\cdot 1}}$$

$$= \frac{1}{(n+1)\cdot n}$$

(lists of #'s)

(n^{th} # in the list)

Limits of Sequences:

as n gets big, where does a_n go?

Theorem: Bring limits inside

If $\lim\limits_{n\to\infty} a_n = L$

(5) and if f is continuous at L

Then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L)$

Eg:

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$$

$$= \sin(0)$$

$$= 0.$$

as $n \to \infty$

$\frac{\pi}{n} \to 0$

and $\sin(x)$ is continuous at $x=0$

lists of #'s

Squeeze Theorem (for Sequences)

If $a_n \le b_n \le c_n$ (for all large enough n)

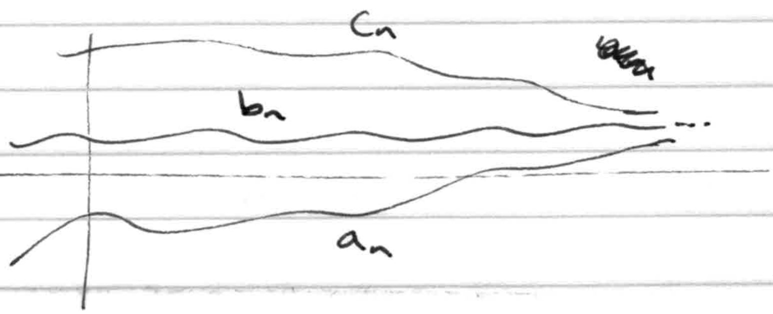
and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

(10)

Then $\lim_{n\to\infty} b_n = L$

(a sort of "comparison theorem for sequences")

Picture:



Eg: $\lim_{n\to\infty} \frac{(-1)^n}{n}$

Notice: $\boxed{\begin{array}{l} -\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n} \\ \text{AND} \\ \lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}}$

so: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$. by squeeze theorem

Same argument:

If $\lim_{n\to\infty} |a_n| = 0$, Then $\lim_{n\to\infty} a_n = 0$.

Eg: $\lim_{n \to \infty} \frac{n!}{n^n}$

$a_n = \frac{n!}{n^n}$

what to compare this to?

$a_1 = \frac{1}{1}$

try writing out terms, to find a pattern

$a_2 = \frac{1 \cdot 2}{2 \cdot 2}$

(10)

$a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$

Notice there are "paired terms" on top & bottom

$\vdots$

$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n}$

group "paired terms", and think about how they relat

too much going on. how can we simplify?

NOTICE: this is less than 1

so: $a_n \leq \frac{1}{n}$

also: $0 < a_n$

so: $0 < a_n \leq \frac{1}{n}$

$\quad\;\; \downarrow \qquad\quad \downarrow$

$\quad\;\; 0 \qquad\quad\; 0$

$\boxed{\text{the sequence converges to 0 by the squeeze theorem}}$

Series Day 1

You can also use a comparison to show that a $\boxed{\text{list of numbers}}$ diverges to ∞

the n^{th} term of

a sequence

Eg: $\lim\limits_{n\to\infty} \frac{n!}{2^n} = ?$

Write out some terms

$a_1 = \frac{1}{2}$

$a_2 = \frac{1\cdot 2}{2\cdot 2}$ Notice $\frac{2}{2} = 1$

(10) $a_3 = \frac{1\cdot 2\cdot 3}{2\cdot 2\cdot 2}$ Notice $\frac{3}{2} > 1$

$a_4 = \frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 2\cdot 2\cdot 2}$ Notice $\frac{4}{2} > 1$

$\vdots$

$$a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot (n-1)\cdot \boxed{n}}{2\cdot 2\cdot 2\cdot 2\cdot \cdots \cdot 2\cdot \boxed{2}}$$

always $\frac{1}{2}$

always greater than 1

$\boxed{\text{want}}$ to compare to this

So NOTICE

$$a_n > \frac{1}{2}\cdot \boxed{\frac{n}{2}} \to \infty$$

so $\lim\limits_{n\to\infty} \frac{n!}{2^n} = \infty$

11.2 Infinite Sums (Series).

[redacted]

[redacted]

want: $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^n}{n!} + \cdots$

(5min)

Define: a series is the $\boxed{\text{SUM}}$

of an infinite ordered list of numbers

Realistically, an infinite sum ~~series~~ must be a limit

our Goal: e^2 is the limit of the sequence

$$\left\{ \boxed{1}, \boxed{1+2}, \boxed{1+2+\frac{2^2}{2!}}, \cdots \right\}$$

$$e^2 = \lim_{n\to\infty}\left(\boxed{1+2+\cdots+\frac{2^n}{n!}}\right)$$

More generally:

[redacted]

For short, we write the sum [redacted]

$$a_1 + a_2 + a_3 + \dots + a_n + \cdots$$

as $\displaystyle\sum_{n=1}^{\infty} a_n$ ← This is a $\boxed{\text{recipe}}$.
START with $n=1$,
and add the terms a_n
Keep going until ∞
(never stop).

(5)

To compute an infinite sum
(or see that it diverges)

We use the limit of
the sequence of "partial sums"

Given:

[redacted] $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

Define:

[redacted]

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$$S_n = \left[a_1 + a_2 + a_3 + \cdots + a_n\right] = \sum_{i=1}^{n} a_i$$

So: $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left[a_1 + a_2 + \dots + a_n\right] = \lim_{n\to\infty} S_n$

SKIP IN CLASS

Formally **Define**

① $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + \cdots + a_n) = \lim_{n\to\infty} S_n$$

② If $\lim_{n\to\infty} S_n = L$ ⟵ a real finite #

Then $\sum_{n=1}^{\infty} a_n = L$ convenges

(no min)

③ If $\lim_{n\to\infty} S_n$ DIVERGES

then $\sum_{n=1}^{\infty} a_n$ divenges.

(we'll come back to e^2 after exam 2)

Most important series:
the geometric series

Eg: $\sum 5\cdot\left(\frac{1}{2}\right)^{n-1} = 5 + \frac{5}{2} + \frac{5}{2^2} + \frac{5}{2^3} + \cdots$

(10) first term $= 5$

common ratio $= \frac{1}{2}$

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{1}{2} \quad \text{for } \underline{\text{all}}\ n.}$$

Define: geometric series has the form

important: this makes $a_1 = a$

$$\sum_{n=1}^{\infty} a\cdot r^{\boxed{n-1}} = \underset{\boxed{\text{first term}}}{a} + a\cdot \underset{\boxed{\text{common ratio}}}{r} + a\cdot r^2 + a\cdot r^3 + \cdots + \underset{a_n}{a\cdot r^{n-1}} + \cdots$$

NOTE: <u>diverges</u> if $|r| = 1$

[$a + a + a + \cdots$ diverges if $a \neq 0$.
$a - a + a - a + \cdots$]

Eg: $\sum_{n=1}^{\infty} 2^n 3^{1-n}$ ← NOT obviously geometric.

$= 2 + 2\left(2\cdot\frac{1}{3}\right) + 2\left(2\cdot\frac{1}{3}\right)\left(2\cdot\frac{1}{3}\right) + \cdots$

$= 2 + 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 + \cdots$

first term

common ratio $r = \frac{2}{3}$

WANT

$\sum_{n=1}^{\infty} a \cdot r^{n-1}$

first term

common ratio

(10) $= \sum_{n=1}^{\infty} \underset{a}{2}\left(\underset{r}{\frac{2}{3}}\right)^{\boxed{n-1}}$

alternate method:

$$\sum_{n=1}^{\infty} 2^n \cdot 3^{1-n} = \sum_{n=1}^{\infty} 2 \cdot 2^{n-1} \cdot 3^{-1(n-1)}$$

$$= \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$$

Questions: ① Does this converge to a real #?

② If so, to what #?

Nice about Geometric series: we can answer $\boxed{\text{BOTH}}$ questions

Remember: geometric series DIVERGE if $|r|=1$

Useful Fact (we'll see why next time)

If $|r| \neq 1$, then the geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1}$

has $S_n = a + a \cdot r + \dots + a \cdot r^{n-1} = \dfrac{a - a \cdot r^n}{1-r}$

partial sum

So: [redacted]

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \lim_{n\to\infty} \left(\frac{a - a \cdot r^n}{1-r} \right)$$

---

(10) when $r = \frac{2}{3}$ & $a = 2$

$$\sum_{n=1}^{\infty} 2 \cdot \left(\frac{2}{3}\right)^{n-1} = \lim_{n\to\infty} \left(\frac{2 - 2 \cdot \left(\frac{2}{3}\right)^n}{1 \text{ [redacted] } - \frac{2}{3}} \right) = \frac{2}{1-\frac{2}{3}} = 6$$

$\left(2 \cdot \left(\frac{2}{3}\right)^n \to 0\right)$, $\quad 1-\frac{2}{3} = \frac{1}{3}$

$$\boxed{\text{In fact, } \underline{\text{whenever}}\ |r| < 1 \qquad \sum_{n=1}^{\infty} a \cdot r^{n-1} = \lim_{n\to\infty} \left(\frac{a - a \cdot r^n}{1-r} \right) = \underbrace{\frac{a}{1-r}}_{\text{converges}}}$$

Notice Also when $|r| > 1$

$$\lim_{n \to \infty} \frac{a - a \cdot r^n}{1 - r} \quad \text{diverges}$$

Summary

If $|r| < 1$

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r} \qquad \text{converges}$$

If $|r| \geq 1$

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} \qquad \text{DIVERGES}$$

11.2 - Series II

A series is the $\boxed{\text{sum}}$ of an ordered list of numbers

(5) A series is the limit of its $\boxed{\text{partial sums}}$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\left(a_1 + \cdots + a_n\right) = \lim_{n\to\infty} S_n$$

---

series

To find what a sum equals you need a formula for S_n

last time & later today

1. $\boxed{\text{geometric series}}$ have nice $\boxed{\text{partial sums}}$ S_n

do this first

2. [redacted] Some [redacted] partial sums $\boxed{\text{"telescope"}}$ [redacted] into nice formulas

fold up neatly

Eg: $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{\text{the whole sum}} = \lim_{n\to\infty} \underbrace{\boxed{S_n}}_{\text{the partial sum}}$ ← the Sum [redacted] of the first n terms

(10)

REWRITE

① [redacted] $S_n = \sum_{i=1}^{n} \boxed{\frac{1}{i(i+1)}}$ so that

later terms will cancel earlier terms.

$$\boxed{\frac{1}{i(i+1)}} = \frac{A}{i} + \frac{B}{i+1} = \boxed{\frac{1}{i} - \frac{1}{i+1}}$$

$$1 = A(i+1) + B \cdot i$$

$i = 0 \Rightarrow A = 1$

$i = -1 \Rightarrow B = -1$

② write out terms until the sum "telescopes"

$$S_n = \sum_{n=1}^{\infty} \boxed{\frac{1}{i} - \frac{1}{i+1}}$$

$$= \left[\begin{array}{ll} \frac{1}{1} - \cancel{\frac{1}{2}} & \leftarrow i=1 \\ + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} & \leftarrow i=2 \\ + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} & \leftarrow i=3 \\ \vdots & \\ + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} & \leftarrow i=n-1 \\ + \cancel{\frac{1}{n}} - \frac{1}{n+1} & \leftarrow i=n \end{array} \right]$$

So

$$S_n = 1 - \frac{1}{n+1}$$

③ use the partial sum
to find the whole sum

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{\text{the } \boxed{\text{whole}} \text{ sum}} = \lim_{n\to\infty} \underbrace{S_n}_{\text{the } \boxed{\text{partial}} \text{ sum}} = \lim_{n\to\infty} \underbrace{\left[1 - \frac{1}{n+1}\right]}_{\text{simplified formula for the sum of the first } n \text{ terms}} = 1$$

(5)

$\Rightarrow$ $\boxed{\text{the whole sum} = 1}$

To make the partial sum telescope,
use
- partial fractions
- log properties
- etc.

When $r \neq 1$, the <u>geometric series</u>

$$\sum_{n=1}^{\infty} \underset{\substack{\uparrow \\ \text{first} \\ \text{term}}}{a} \cdot \underset{\substack{\uparrow \\ \text{common} \\ \text{ratio}}}{r}^{n-1}$$

has nice partial sums

<u>why?</u>

sum of first n terms $= S_n = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1}$

(multiply both sides by r)

(10) $r \cdot S_n = \overset{\text{nothing!}}{} \; a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1} + a \cdot r^n$

(subtracting these 2 equations)

$$S_n - r \cdot S_n = a + 0 + 0 + \cdots + 0 - a r^n$$

$$= a - a \cdot r^n$$

<u>solving for S_n:</u>

$$S_n(1-r) = a - a \cdot r^n$$

$$S_n = \frac{a - a \cdot r^n}{1-r} = \text{sum of first } n \text{ terms of the series } \sum_{n=1}^{\infty} a \cdot r^{n-1}$$

we used this last time to show that

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r} \quad \text{whenever } |r| < 1$$

AND $\sum^{\infty} a \cdot r^{n-1}$ diverges when $|r| \geq 1$

We mostly can't compute # values
for non-geometric, non-telescoping series

Instead we will test infinite sums ← series
to see if they converge or diverge

Our first

Test for Divergence

(10) **If** $\lim\limits_{n\to\infty} a_n \neq 0$

(or if $\lim\limits_{n\to\infty} a_n$ doesn't exist)

Then $\sum\limits_{n=1}^{\infty} a_n$ diverges

If $a_n \not\to 0$,
then the partial sums don't settle down

If partial sums don't settle down
then the series diverges

Eg: What does $\sum\limits_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ do? ← a_n

Note: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2}{5n^2+4} = \lim\limits_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{1}{\left(5+\frac{4}{n^2}\right)} = \frac{1}{5}$

(So $\sum \frac{n^2}{5n^2+4}$ is essentially the sum of ∞-many $\frac{1}{5}$'s
Thus the sum diverges to ∞)

So the series diverges by the "test for divergence"

We are well on our way
to the land of abstract nonsense

But first, one more cute and very concrete example.

Repeating Decimals are fractions!

Eg: $2.\overline{17} = 2.\underbrace{17}\,\underbrace{17}\,\underbrace{17}\cdots$

$$= 2 + \frac{17}{100} + \frac{17}{10000} + \frac{17}{1000000} + \cdots$$

(10) $$= 2 + \underbrace{\frac{17}{10^2} + \frac{17}{10^4} + \frac{17}{10^6} + \cdots}_{\text{a geometric series!}}$$

First term: $\frac{17}{10^2}$

Common Ratio:

$$r = \frac{a_2}{a_1} = \frac{\frac{17}{10^4}}{\frac{17}{10^2}} =$$

$$= \frac{17}{10^4} \cdot \frac{10^2}{17} = \frac{1}{10^2}$$

$$\left(\text{check } r = \frac{a_3}{a_2} = \frac{\frac{17}{10^6}}{\frac{17}{10^4}} = \frac{1}{10^2} \checkmark\right)$$

$$= 2 + \sum_{n=1}^{\infty} \left(\frac{17}{10^2}\right) \cdot \left(\frac{1}{10^2}\right)^{n-1}$$

because $|r|<1$

$$= 2 + \frac{\frac{17}{10^2}}{1 - \frac{1}{10^2}} \left(\frac{10^2}{10^2}\right) = 2 + \frac{17}{10^2 - 1}$$

SO $\boxed{2.\overline{17} = 2 + \frac{17}{99}}$