**Instructions:**

- This exam contains 11 pages. When we begin, check you have *one* of each page.
- You will have 70 minutes to complete the exam.
- Please **show all work**, and then **write your answer on the line provided.**
  In order to receive full credit, solutions must be complete, logical and understandable.
- Turn smart phones, cell phones, and other electronic devices off now!

*Academic Honesty:*

By writing my name below, I agree that all the work which appears on this exam is entirely my own.

I will not look at other peoples' work, and I will not communicate with anyone else about the exam.

I will not use any calculators, notes, etc.

I understand that violating the above carries *serious consequences*, both moral and academic.

Printed Name: Key Signature: ______________

Section: ______________

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 10 | 10 | 8 | 9 | 13 | 12 | 6 | 12 | 10 | 100 |
| Score: | | | | | | | | | | | |

1. [10 points] A small colony of deep sea worms is growing at a constant rate per worm. When you first observe the colony, you estimate that it contains 100 worms. After 5 months, you estimate that the colony contains 300 worms.

   (a) Find a formula for the population as a function of $t$ months since your first observation.

$$P(t) = 100 \cdot e^{kt}$$

Know

$$p(5) = 300 = 100 \cdot e^{(k \cdot 5)}$$

$$3 = e^{5k}$$

$$\ln(3) = 5 \cdot k$$

3pt

$$k = \frac{\ln(3)}{5}$$

$$\boxed{P(t) = 100 \cdot e^{\left(\frac{\ln(3)}{5} \cdot t\right)}}$$

2pt

   (b) How long must you wait until the population has doubled?

want $t$ so that

2pt

$$200 = 100 \cdot e^{\left(\frac{\ln(3)}{5} t\right)}$$

$$2 = e^{\left(\frac{\ln(3)}{5} t\right)}$$

$$\ln(2) = \frac{\ln(3)}{5} t$$

3pt

$$\boxed{\frac{5 \cdot \ln(2)}{\ln(3)} = t}$$

2. Evaluate the following limits. Be sure to **show your work** and **circle your answer**.

(a) [5 points] $\lim_{x\to 3^+} \frac{x^2+4x+3}{x^2-2x-3}$

$\boxed{\text{CANNOT plug in 3: } 3^2 - 2\cdot 3 - 3 = 9-6-3 = 0}$

2pt →

$$= \lim_{x\to 3^+} \frac{(x+3)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}}$$

$$= \lim_{x\to 3^+} \frac{x+3}{x-3}$$

2pt →

$$\boxed{\underline{\text{think}}: \approx \frac{\text{just over } 6}{\text{small pos}} \approx \underline{\text{BIG}} \text{ pos}}$$

1pt → $\boxed{= \infty}$

−1 pt if you leave out the "$\lim_{x\to 3^+}$" or "$\lim_{x\to 1^-}$"

(b) [5 points] $\lim_{x\to 1^-} \frac{x-1}{x^2+x-2}$

1pt → $\boxed{\text{CANNOT plug in 1: } 1^2 + 1 - 2 = 0}$

2pt →

$$= \lim_{x\to 1^-} \frac{\cancel{(x-1)}\cdot 1}{\cancel{(x-1)}(x+2)}$$

2pts ONLY if rewrite as $\frac{0}{x+2}$ or $(x+2)$ only

$$= \lim_{x\to 1^-} \frac{1}{x+2}$$

(can plug in 1 by limit law 5)

2pt → $\boxed{= \frac{1}{3}}$



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3. Evaluate the following limits. Be sure to **show your work** and **circle your answer**.

(a) [5 points] $\lim_{x\to\infty} \dfrac{20+3x^2}{x^3-x}$

$$= \lim_{x\to\infty} \frac{x^2\left(\frac{20}{x^2}+3\right)}{x^3\left(1-\frac{1}{x^2}\right)} \quad \frac{20}{x^2}\to 0, \ \frac{1}{x^2}\to 0$$

2pt

$$= \lim_{x\to\infty} \frac{1\cdot 3}{x\cdot 1}$$

1pt

$$= \lim_{x\to\infty} \frac{3}{x}$$

2pt

$$= 0$$

(fastest going)

(b) [5 points] $\lim_{x\to\infty} \dfrac{20-9x^2}{3x^2+4x}$

$$\cancel{= \lim_{x\to\infty} \frac{x^3\left(\frac{3}{x}-\frac{9}{x}\right)}{x^3\left(3+\frac{4}{x^2}\right)} = \frac{-9}{3} = -3}$$

$$= \lim_{x\to\infty} \frac{x^2\left(\frac{20}{x^2}-9\right)}{x^2\left(3+\frac{4}{x}\right)} \quad \frac{20}{x^2}\to 0, \ \frac{4}{x}\to 0$$

3pts

$$= \frac{-9}{3}$$

$$= -3.$$

2pts

4. [8 points]

(a) $\lim_{x \to 0^-} e^{-1/x}$

think:

$\approx e^{-1/\text{tiny } \cancel{\text{neg}}}$

$\approx e^{1/\text{tiny } \underline{\text{pos}}}$

$\approx e^{\text{BIG pos}}$

$\approx$ BIG

2pts

2pts

$= \infty$

(b) $\lim_{x \to \infty} \left[ \frac{\sin(x)}{e^x} \right]$

think

$\approx \frac{\sin(\text{HUGE})}{e^{\text{HUGE}}}$

$\approx \frac{\#\text{between } -1 \text{ \& } 1}{\text{BIG}}$

$\approx$ tiny

2pts for work (REQUIRED)

$= 0$

2pts for answer

$-1 \leq \sin(x) \leq 1$

so

$\frac{-1}{e^x} \leq \frac{\sin(x)}{e^x} \leq \frac{1}{e^x}$

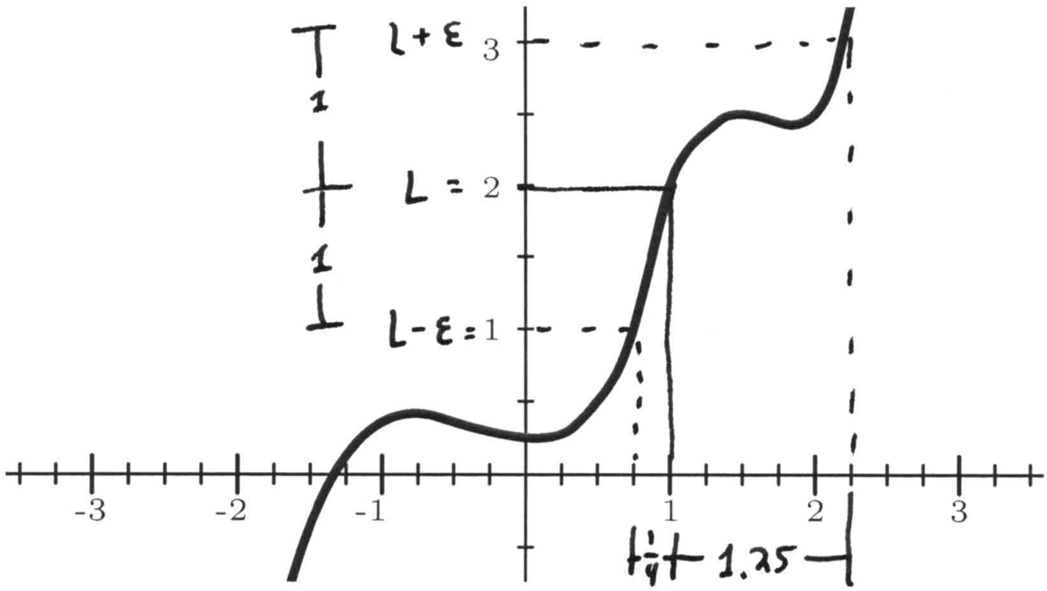
$\lim_{x \to \infty} \frac{-1}{e^x} = 0$

$\lim_{x \to \infty} \frac{1}{e^x} = 0$

so

$$\boxed{\lim_{x \to \infty} \frac{\sin(x)}{e^x} = 0}$$

5. (a) [5 points] Suppose that $g(x)$ is defined using the above graph.




3pt for picture

Find a distance $\delta$ so that if $x$ is within $\delta$ units of 1, then $g(x)$ is within 1 unit of 2. You must support your answer by **what you draw** in the figure.

2pt for δ

$\boxed{\text{let } \delta = 0.25.}$

If $x$ is within 0.25 of 1

then $g(x)$ is within 1 unit of 2

(b) [4 points] Suppose that $f(x) = 4x + 2$. Find a number $\delta$ so that if $|x - 1| < \delta$ then $|f(x) - 6| < 2$.

want $|f(x) - 6| < 2$

1pt

$|(4x+2) - 6| < 2$

2pt

$|4x - 4| < 2$

$|4| \cdot |x-1| < 2$

$|x-1| < \frac{2}{4} = \frac{1}{2}$

$\boxed{\text{let } \delta = \frac{1}{2}}.$ 1pt

(If ~~f(x)~~ $x$ is within $\frac{1}{2}$ of 1, then $f(x)$ is within 2 of 6).



3:05

6. The definition and meaning of the derivative

(a) [4 points] Write down the limit definition of the derivative of the function $f(x)$.

$$f'(x) = \underbrace{\lim_{h \to 0}}_{(2pts)} \underbrace{\frac{f(x+h) - f(x)}{h}}_{2pts}$$

(b) [4 points] Explain why $f'(x)$ describes the instantaneous rate of change of $f$ at $x$

$$f'(x) = \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}$$

avg. rate of change of f NEAR $x$

over a shorter & shorter interval.

(c) [5 points] Let $f(x) = x^2 + 2x$. Find $f'(3)$ using the limit definition of the derivative.

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

2pts

$$f(3+h) = (3+h)^2 + 2 \cdot (3+h)$$
$$= 9 + 6h + h^2 + 6 + 2h$$
$$= 15 + 8h + h^2$$
$$f(3) = 3^2 + 2 \cdot 3 = 9 + 6 = 15$$

$$= \lim_{h \to 0} \frac{(\cancel{15} + 8h + h^2) - (\cancel{15})}{h}$$

2pts

$$= \lim_{h \to 0} \frac{8h + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(8+h)}{\cancel{h}} = 8.$$

1pt



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7. [12 points] From this question onward, you may use the derivative rules.
You must **show major steps** for credit.

Suppose your position at time $t$ is given by $s(t) = t^4 - \frac{1}{t} + 2t + 1$.

(a) Find your velocity at time $t = 1$.

$$v(t) = s'(t) = \frac{d}{dt}\left[t^4 - t^{-1} + 2t + 1\right]$$

2pt

$$= 4\cdot t^3 - \boxed{(-1)t^{-2}} + 2$$

2pt rest

$$= 4t^3 + t^{-2} + 2$$

2pt

$$v(1) = 4\cdot 1^3 + \frac{1}{1^2} + 2 = 4+1+2 = 7$$

(b) Find your acceleration at time $t = 1$.

$$a(t) = s''(t) = \frac{d}{dt}\left[4\cdot t^3 + t^{-2} + 2\right]$$

2pt

$$= \boxed{4\cdot 3t^2} + \boxed{(-2)t^{-3}} + 0$$

2pt

$$= 12t^2 - \frac{2}{t^3}$$

2pt

$$a(1) = 12\cdot 1^2 - \frac{2}{1^3} = 12-2 = 10$$



2:10

8. (a) [6 points] Let $f(x) = e^x \cdot \tan(x)$. Find the derivative of $f$.

$$f'(x) = \frac{d}{dx}\left[ e^x \cdot \tan(x) \right]$$

4pts

$$= e^x \cdot \frac{d}{dx}\left[\tan(x)\right] + \tan(x) \cdot \frac{d}{dx}\left[e^x\right]$$

$$= e^x \cdot \sec^2(x) + \tan(x) \cdot e^x$$

2pts

$$= e^x \cdot \left(\sec^2(x) + \tan(x)\right)$$

9. [12 points] Let $f(x) = \frac{x}{1+x^2}$.

$$\left(\frac{t}{b}\right)' = \frac{bt' - tb'}{b^2}$$

(a) Find the instantaneous rate of change of $f(x)$ at 2.

$$f'(x) = \frac{d}{dx}\left[\frac{x}{1+x^2}\right]$$

2pt

$$= \frac{(1+x^2)\cdot\frac{d}{dx}[x] - x\cdot\frac{d}{dx}[1+x^2]}{(1+x^2)^2}$$

2pt

$$= \frac{(1+x^2)\cdot 1 - x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

2pt

$$= \frac{1-x^2}{(1+x^2)^2}$$

$$f'(2) = \frac{1-2^2}{(1+2^2)^2} = \frac{1-4}{(1+4)^2} = \frac{-3}{5^2} = \frac{-3}{25}$$

(b) Find the equation of the line tangent to $f(x)$ at $(2, \frac{2}{5})$.

2pt

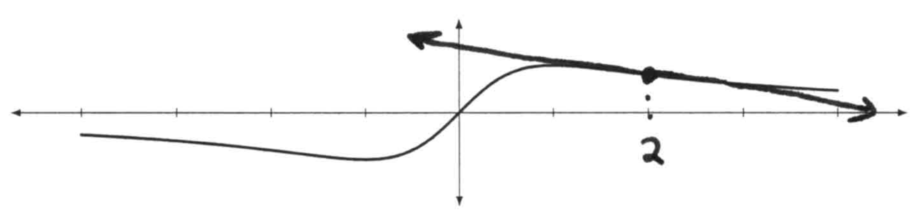
$$y = m(x-2) + f(2)$$

2pt

$$y = \frac{-3}{25}(x-2) + \frac{2}{5}$$

(c) Sketch the line tangent to $f(x)$ at 2.

2pt



10. (a) [5 points] Let $y = 5(3x + 1)^4$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[5 \cdot (3x+1)^4\right]$$

$$= 5 \cdot \frac{d}{dx}\left[(3x+1)^4\right]$$

3pt $$= 5 \cdot 4\,(3x+1)^3 \cdot \frac{d}{dx}\left[3x+1\right]$$

2pt $$= 20 \cdot (3x+1)^3 \cdot 3$$

$$= 60 \cdot (3x+1)^3$$

(b) [5 points] Let $f(x) = \sqrt{x^4 - x}$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[(x^4 - x)^{1/2}\right]$$

3pt $$= \frac{1}{2} \cdot (x^4 - x)^{-1/2} \cdot \frac{d}{dx}\left[x^4 - x\right]$$

$$= \frac{1}{2} \cdot (x^4 - x)^{-1/2} \cdot (4x^3 - 1)$$

2pt $$= \frac{4x^3 - 1}{2\sqrt{x^4 - x}}$$



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