Newton's Method

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Math 1131, Class 25

Solving Equations

Many science and engineering problems can be phrased as

$$\text{“Solve} \quad f(x) = 0 \quad \text{”}$$

Strategy: Make a first guess,
and use linear approximations to iteratively improve it.

An Example

Question

What is $\sqrt[3]{2}$?

Solving

$$x^3 = 20$$

is the same as solving

$$x^3 - 20 = 0$$

Let $f(x) = x^3 - 20$. Then $f'(x) = 3x^2$.

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Where is this **linearization** zero?

$$\begin{aligned} L_{a_0}(x) &= 0 \\ f'(a_0)\Big(x - a_0\Big) + f(a_0) &= 0 \\ x &= a_0 - \frac{f(a_0)}{f'(a_0)} \end{aligned}$$

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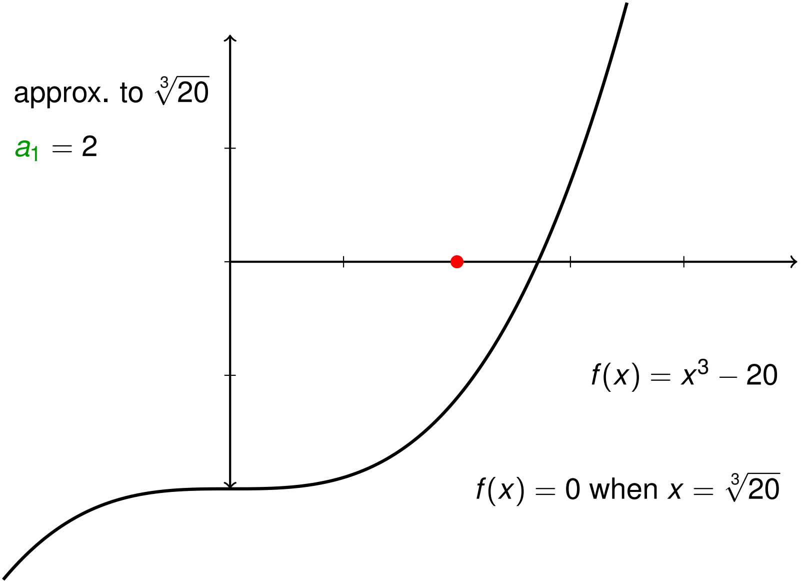
Our next **guess** is that $f(x) \approx 0$ at $a_1 = a_0 - \dfrac{f(a_0)}{f'(a_0)}$

"Rinse and repeat"

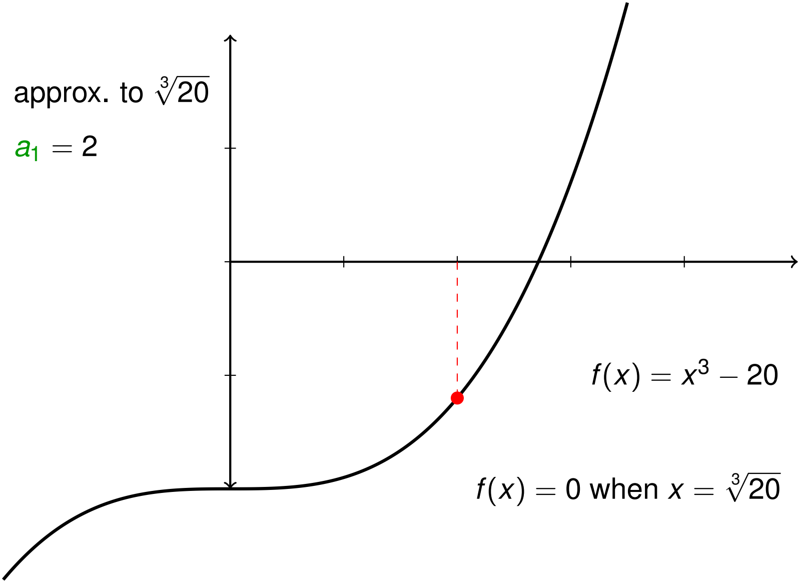
Newton's Method



Newton's Method

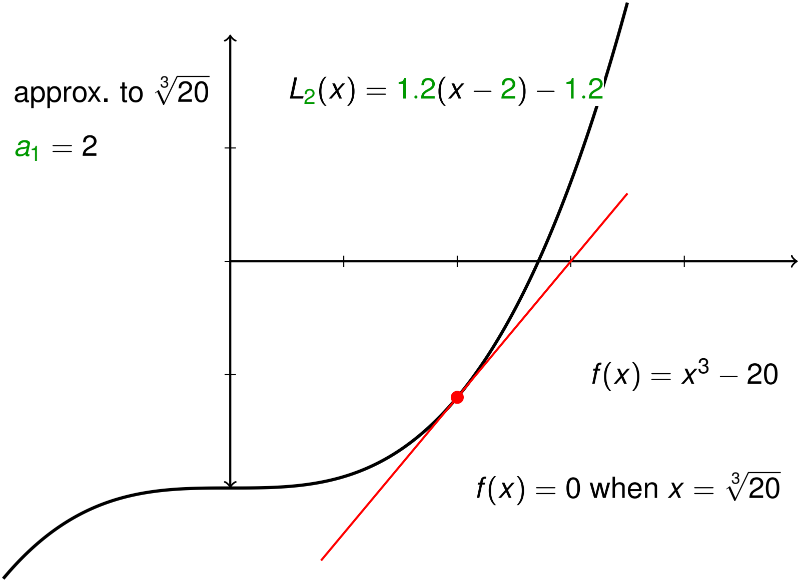
approx. to $\sqrt[3]{20}$

$a_1 = 2$



$f(x) = x^3 - 20$

$f(x) = 0$ when $x = \sqrt[3]{20}$

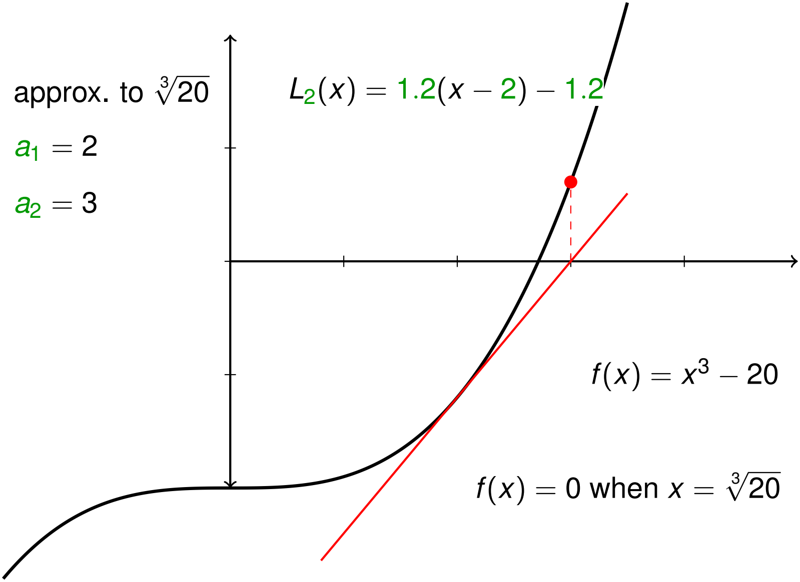

approx. to $\sqrt[3]{20}$
$a_1 = 2$
$L_2(x) = 1.2(x-2) - 1.2$
$f(x) = x^3 - 20$
$f(x) = 0$ when $x = \sqrt[3]{20}$

Newton's Method

approx. to $\sqrt[3]{20}$

$a_1 = 2$

$a_2 = 3$



Newton's Method

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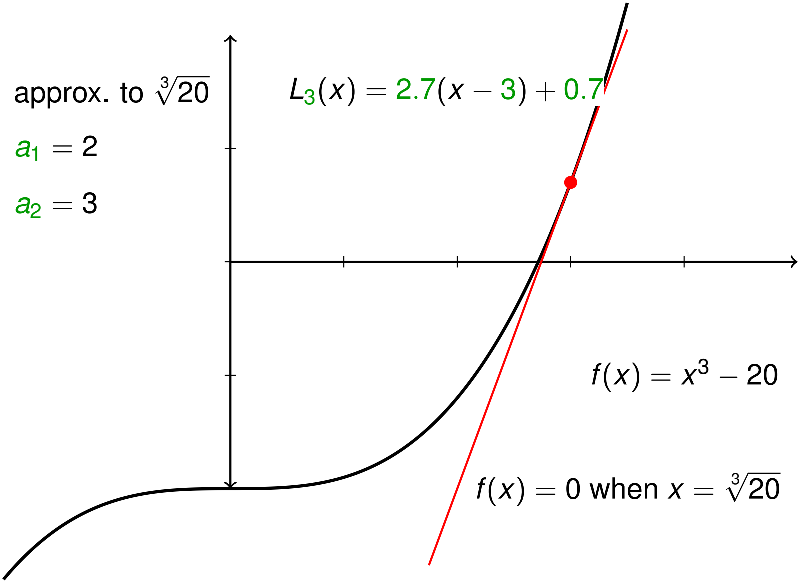


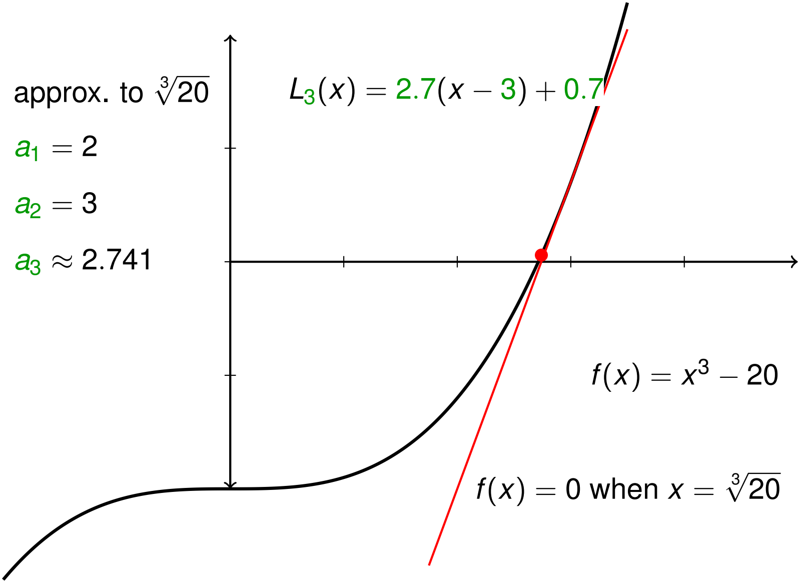
Newton's Method

approx. to $\sqrt[3]{20}$

$a_1 = 2$

$a_2 = 3$

$a_3 \approx 2.741$



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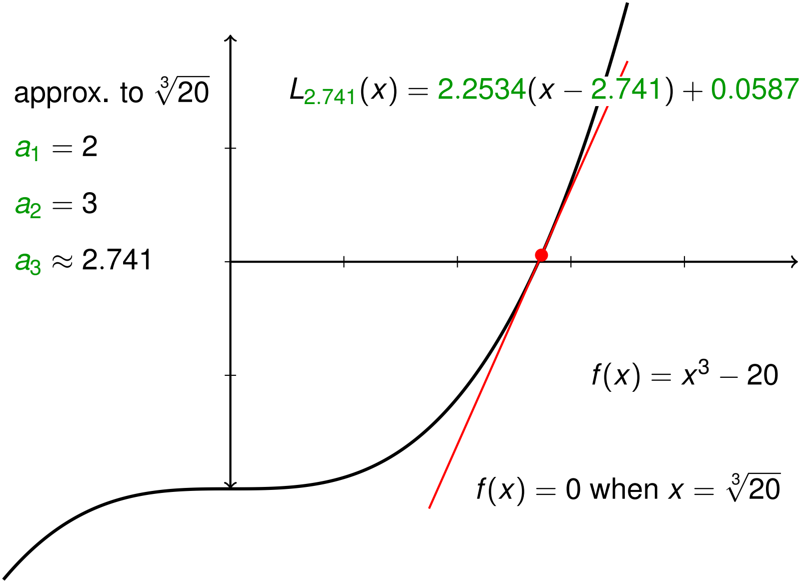

Newton's Method

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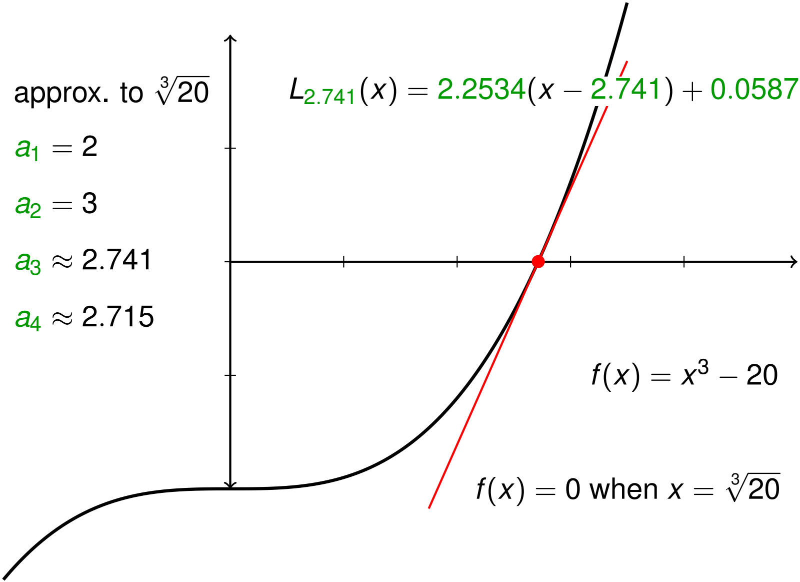

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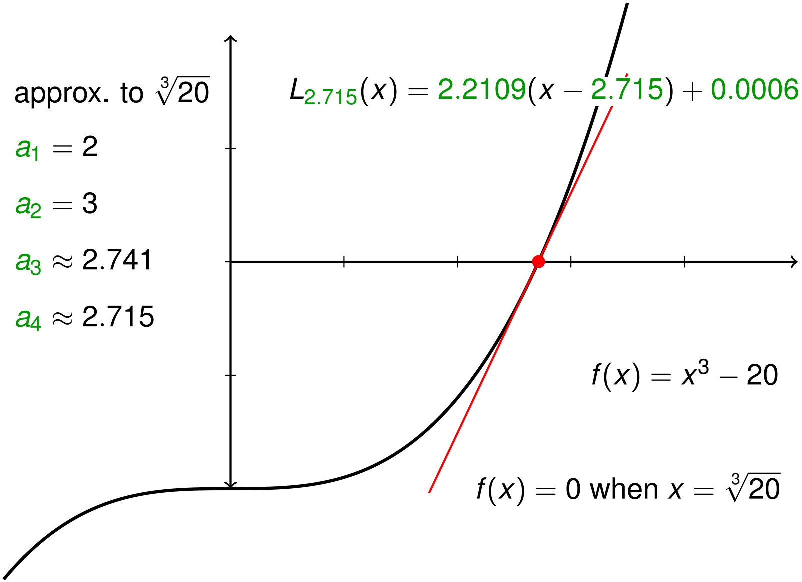

approx. to $\sqrt[3]{20}$

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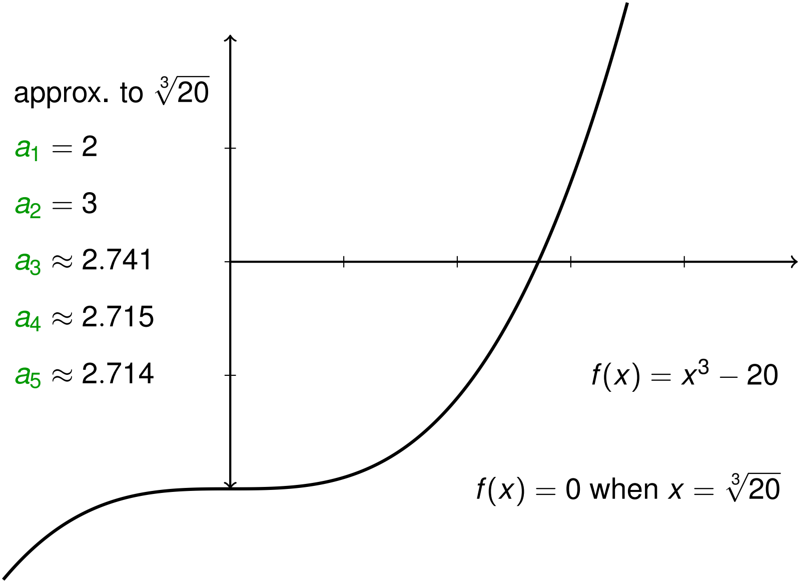
$a_5 \approx 2.714$



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$\sqrt[3]{20} = 2.7144\ldots$