Instructions:

- This exam contains 14 pages. When we begin, check you have *one* of each page.
- You will have 2 hours to complete the exam.
- Please **show all work**, and then **write your answer on the line provided.** In order to receive full credit, solutions must be complete, logical and understandable.
- Turn smart phones, cell phones, and other electronic devices off now!

Academic Honesty:
By writing my name below, I agree that all the work which appears on this exam is entirely my own.
I will not look at other peoples' work, and I will not communicate with anyone else about the exam.
I will not use any calculators, notes, etc.
I understand that violating the above carries *serious consequences*, both moral and academic.

Printed Name: Key Signature: ____________

Section: ____________

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	12	12	12	12	10	10	10	12	12	12	12	12	12	150
Score:														

1. (a) [4 points] Let $f(x) = \tan(x^2)$. Find $f'(x)$.

$$f'(x) = \sec^2(x^2) \cdot \frac{d}{dx}[x^2]$$

$$= 2x \cdot \sec^2(x^2)$$

← 4 pt

(b) [4 points] Let $f(x) = \sqrt{\ln(x)}$. Find $f'(x)$ and simplify completely.

$$f'(x) = \frac{d}{dx}\left[(\ln(x))^{\frac{1}{2}} \right]$$

$$= \frac{1}{2} \cdot (\ln(x))^{-\frac{1}{2}} \cdot \frac{d}{dx}[\ln(x)]$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\ln(x)}} \cdot \frac{1}{x}$$

← 4 pt

(c) [4 points] Let $y = e^{2x+4}$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = e^{2x+4} \cdot 2$$

← 4 pt

2. [12 points] Compute the following limits, showing your work.

(a) Compute the limit $\lim_{x\to 2^+} \dfrac{x^2 - x - 2}{x - 2}$

cannot plug in 2

$$= \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}}$$

← 2pt

$$= \lim_{x\to 2^+} (x+1)$$

$$= 3$$

← 2pt

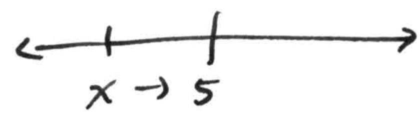
(b) Compute the limit $\lim_{x\to 5^-} \dfrac{1}{x - 5}$

cannot plug in 5

2pt

as $x \to 5^-$
$x - 5$ is ① small ② negative

so $\dfrac{1}{x-3}$ is ① big ② Negative



2pt →

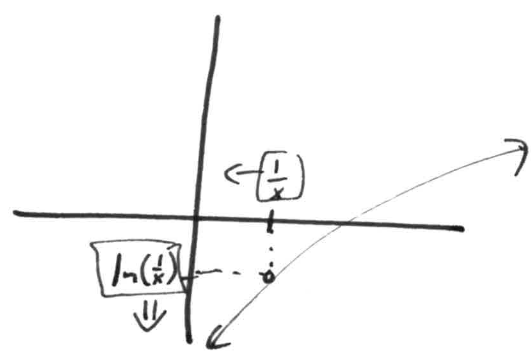
$$= -\infty$$

(c) Compute the limit $\lim_{x\to\infty} \ln\left(\dfrac{1}{x}\right)$

1pt →

as $x \to \infty$,
$\dfrac{1}{x}$ is ① small ② positive

so $\left(\frac{1}{x}\right) \to 0^+$



$$= \lim_{\left(\frac{1}{x}\to 0^+\right)} \ln\left(\tfrac{1}{x}\right) = -\infty$$

1pt (picture)

↑
2pt

3. (a) [6 points] Compute the limit $\lim_{x\to\infty} \frac{2x+x^2}{3x^2-7}$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{\left(\frac{2}{x}+1\right)}{\left(3 - \frac{7}{x^2}\right)}$$ ← 3pt

$\frac{2}{x} \to 0$, $\frac{7}{x^2} \to 0$

$$= \frac{1}{3}$$ ← 3pt

(b) [6 points] Use L'Hopital's rule to compute the limit $\lim_{x\to\infty} \frac{x\ln(x)}{x^2}$ $\begin{matrix}\to \infty \\ \to \infty\end{matrix}$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x}$$

3pt → $$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

3pt → $$= 0$$

OR

$$\overset{H}{=} \lim_{x\to\infty} \frac{x\cdot\frac{1}{x} + 1\cdot\ln(x)}{2x}$$ ← 2pt

$$= \lim_{x\to\infty} \frac{1+\ln(x)}{2x} \begin{matrix}\to \infty \\ \to \infty\end{matrix}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2}$$ ← 2pt

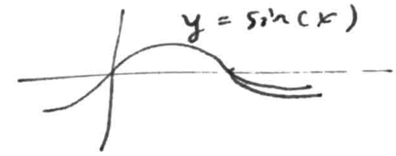
$$= 0$$ ← 2pt

4. [12 points] Let $f(x) = x\cos(x) + x$

(a) Find an equation for the line tangent to the curve $y = x\cos(x) + x$ at $a = 0$.

2pt

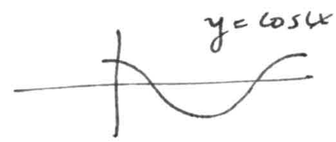
$$f'(x) = \underbrace{x \cdot \frac{d}{dx}[\cos(x)] + \cos(x) \cdot \frac{d}{dx}[x]}_{\text{product rule.}} + \frac{d}{dx}[x]$$



$$= x \cdot (-\sin(x)) + \cos(x) + 1$$

2pt

$$f'(0) = \underbrace{0 \cdot (-\sin(0))}_{0} + \underbrace{\cos(0)}_{1} + 1$$



$$= 1 + 1 = 2$$

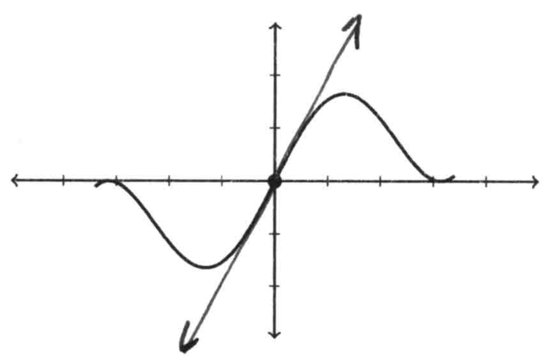
$$f(0) = 0 \cdot \cos(0) + 0 = 0$$

$$y = m(x - x_1) + y_1$$

2pt → $y = 2(x-0) + 0 \quad = 2x$

(b) Sketch the line tangent to the curve at the point $(0,0)$.

2pt



(c) Find the linearization $L(x)$ of $f(x)$ at $a = 0$, and use it to approximate $f(0.2)$.

2pt → $L(x) = 2(x-0) + 0$

$$f(0.2) \approx L(0.2) = 2(0.2 - 0) + 0$$

$$= 0.4$$

2pt

5. [10 points] Suppose that a population of bacteria is growing in a petri dish. Suppose also that the first time you look at the dish you count 20 bacteria, and that you count 200 bacteria in the dish 2 hours later.

(a) Find a formula for the population as a function of the number of hours t since your first measurement.

$$P(t) = 20 \cdot e^{kt}$$

find t s.t. $\rightarrow$ $P(2) = 200 = 20 \cdot e^{k \cdot 2}$

1pt

$$10 = e^{k \cdot 2}$$

$$\ln(10) = k \cdot 2$$

$$k = \frac{\ln(10)}{2}$$

2pt

2pt

$$\boxed{P(t) = 20 \cdot e^{\left(\frac{\ln(10)}{2} t\right)}}$$

(b) How much time is required for the population to triple in size?

find t s.t.

2pt

$$60 = 20 \cdot e^{\left(\frac{\ln(10)}{2} t\right)}$$

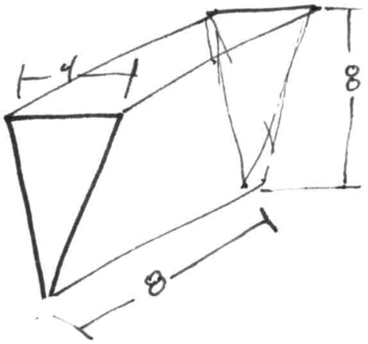
$$3 = e^{\frac{\ln(10)}{2} t}$$

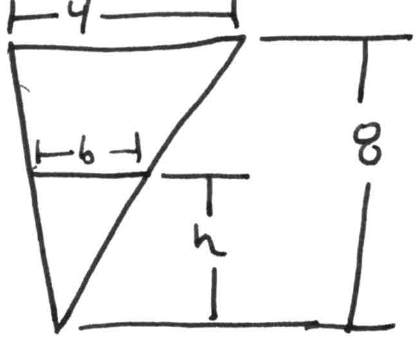
$$\ln(3) = \frac{\ln(10)}{2} t$$

3pt

$$\boxed{t = \frac{2 \cdot \ln(3)}{\ln(10)}}$$

6. [10 points] Suppose there is a 8' long water trough shaped as a triangular prism whose cross-section is an inverted triangle ∇ which is 4' wide across the top, and which is 8' tall. If the tank is being being filled with water at a constant rate of 160 ft^3/s, how fast is the height changing at the time when the water is half way up the tank (4' above the bottom)?





1pt sketch

Relate Volume & height

1pt $V = \underbrace{\left(\frac{1}{2}\cdot b\cdot h\right)}_{\text{Cross section area}} \cdot 8$ ← length

Relate base & height

$$\frac{b}{h} = \frac{4}{8}$$
$$4h = 8b$$
$$b = \frac{h}{2}$$ 1pt

2pt $V = \frac{1}{2}\cdot\frac{h}{2}\cdot h\cdot 8 = 2h^2$

Relate the Rates

3pt
$$\frac{d}{dt}[V] = \frac{d}{dt}\left[2h^2\right]$$
$$\frac{dV}{dt} = 2\cdot 2h\cdot\frac{dh}{dt}$$

−1 pt if lose one 2

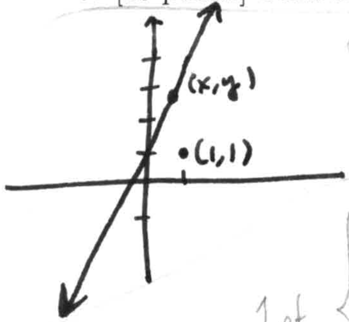
when $h=4$, $\frac{dV}{dt} = 160$

$$160 = 4\cdot 4\cdot\frac{dh}{dt}$$

2pt

$$\frac{dh}{dt} = \frac{160}{4\cdot 4} = \frac{40}{4} = 10$$

7. [10 points] Find the point on $y = 3x + 1$ which is as close as possible to $(1, 1)$.



2pt for setup

Minimize distance subject to constraint

$y = 3x + 1$

1pt

$$\text{dist} = \sqrt{(x-1)^2 + (y-1)^2}$$

$$= \sqrt{(x-1)^2 + (3x+1-1)^2}$$

easier to minimize $(\text{dist})^2$

2pt

$$F(x) = (\text{dist})^2 = (x-1)^2 + (3x)^2$$

$$F(x) = x^2 - 2x + 1 + 9x^2$$

$$F(x) = 10x^2 - 2x + 1 \Leftarrow$$ minimize this

2pt

$$F'(x) = 20x - 2$$

critical #'s.

F' always defined

$F'(x) = 0$ when $20x = 2$

$x = \frac{1}{10}$

distance is minimized when

$$x = \frac{1}{10}$$

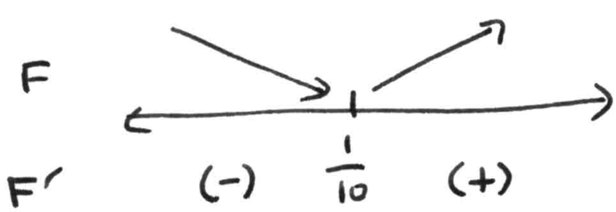
$$y = 3 \cdot \frac{1}{10} + 1$$

$$= \frac{3}{10} + \frac{10}{10}$$

= [redacted] $\frac{13}{10}$

2pt

1pt to check

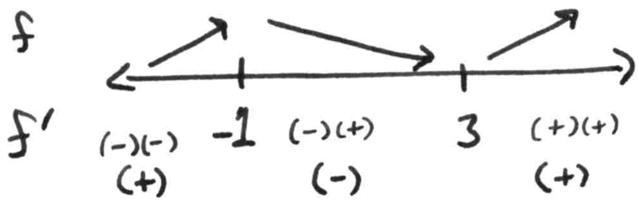


8. [12 points] Let $f(x) = \frac{x^3}{3} - x^2 - 3x + 4$

Find the following if they exist (or write DNE). You must **show all work**.

1. Find the intervals where $f(x)$ is increasing/decreasing. Identify which is which.

$$f'(x) = \frac{3x^2}{3} - 2x - 3 = x^2 - 2x - 3 = (x-3)(x+1)$$



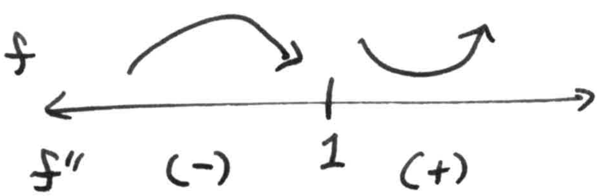
increasing on $(-\infty,-1) \cup (3,\infty)$

decreasing on $(-1,3)$

5/5

2. Find the intervals where $f(x)$ is concave up/down. Identify which is which.

$$f''(x) = 2x - 2 = 2(x-1)$$



concave up on $(1,\infty)$

concave down on $(-\infty,1)$

4/4

3. Find the x value(s) of the local maxima and local minima of f. Identify which is which.

local max at -1

local min at 3

2/2

1pt for each #/interval

1pt each for f' & f''

1pt for each sign chart

4. Find the x value(s) of the inflection points of f.

inflection at 1

1/1

9. [12 points] (a) Compute the general antiderivative for $f(x) = \dfrac{1+x^3}{x^2}$

$$f(x) = \frac{1}{x^2} + \frac{x^3}{x^2} = x^{-2} + x \quad \leftarrow 2pt$$

$$F(x) = \frac{x^{-1}}{-1} + \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} - \frac{1}{x} + C \quad \leftarrow 2pt$$

(b) Suppose that $f''(x) = 6x^2 - 12x + 4$, that $f'(0) = 1$ and that $f(0) = 3$. Find a formula for $f(x)$.

2pt

$$f'(x) = \frac{6x^3}{3} - \frac{12x^2}{2} + 4x + C = 2x^3 - 6x^2 + 4x + C$$

2pt

$$f'(0) = 1 = 2\cdot 0^3 - 6\cdot 0^2 + 4\cdot 0 + C$$

$$C = 1$$

$$f'(x) = 2x^3 - 6x^2 + 4x + 1$$

$$f(x) = \frac{2x^4}{4} - \frac{6x^3}{3} + \frac{4x^2}{2} + x + D$$

2pt

$$= \frac{x^4}{2} - 2x^3 + 2x^2 + x + D$$

$$f(0) = 3 = \frac{0^4}{2} - 2\cdot 0^3 + 2\cdot 0^2 + 0 + D$$

$$D = 3$$

2pt

$$\boxed{f(x) = \frac{x^4}{2} - 2x^3 + 2x^2 + x + 3}$$

10. [12 points] Compute the following integrals.

(a) Compute $\int_1^2 x\sqrt{\boxed{x^2-1}}\,dx$ $\frac{du}{2}$ u

3pt

$$\left(\begin{array}{l|l} u = x^2-1 & x=1 \Rightarrow u = 1^2-1 = 0 \\ \frac{du}{dx} = 2x & x=2 \Rightarrow u = 2^2-1 = 3 \\ \frac{du}{2} = x\,dx & \end{array} \right)$$

$$= \int_0^3 u^{\frac{1}{2}} \cdot \frac{du}{2}$$

$$= \left[\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 = \left[\frac{u^{\frac{3}{2}}}{3}\right]_0^3$$

3pt

$$= \frac{3^{3/2}}{3} - \frac{0^{3/2}}{3} = \frac{1}{3} \cdot 3^{3/2}$$

(b) Compute $\int_{-1}^{1} (x+2)(x-4)\,dx$

2pt $$= \int_{-1}^{1} [x^2 - 2x - 8]\,dx$$

2pt $$= \left[\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} - 8x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} - 1^2 - 8\cdot 1\right) - \left(\frac{(-1)^3}{3} - (-1)^2 - 8(-1)\right)$$

$$= \left(\frac{1}{3} - 1 - 8\right) - \left(\frac{-1}{3} - 1 + 8\right)$$

2pt $$= \frac{2}{3} - 16$$

2/23

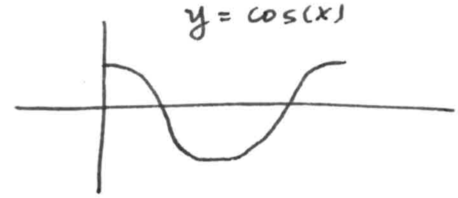
11. [12 points] Compute the following integrals.

(a) Compute $\int_0^{2\pi} \sin\left(\frac{x}{2}\right)\, dx$

2pt

$$\left(\begin{array}{l|l} u = \frac{x}{2} & x = 0 \Rightarrow u = \frac{0}{2} = 0 \\ \frac{du}{dx} = \frac{1}{2} & \\ 2\,du = dx & x = 2\pi \Rightarrow u = \frac{2\pi}{2} = \pi \end{array} \right)$$

$$= \int_0^{\pi} \sin(u) \cdot 2\, du$$



2pt

$$= \Big[-2 \cdot \cos(u) \Big]_0^{\pi}$$

$$= (-2)\cdot\cos(\pi) - (-2)\cdot\cos(0)$$

$$= (-2)(-1) - (-2)\cdot 1 = 2 + 2 = 4$$

2pt

(b) Compute $\int \left[3x^2 + e^{3x}\right] dx$

$$= 3\int x^2\, dx + \int e^{3x}\, dx$$

3pt

$$\left(\begin{array}{l} u = 3x \\ \frac{du}{dx} = 3 \\ \frac{du}{3} = dx \end{array} \right)$$

$$= \cancel{3 \frac{x^3}{3}} + \int e^u \cdot \frac{du}{3}$$

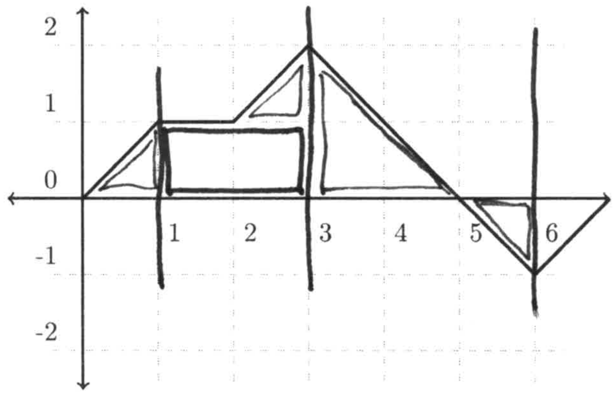
$$= x^3 + \frac{1}{3} e^u + C$$

$$\boxed{= x^3 + \frac{1}{3} e^{3x} + C}$$

3pt

12. [12 points] Suppose that the function $f(x)$ is given by the following graph.

2pt each



Let $A(x) = \int_0^x f(t)\,dt$. Compute the following

(a) $A(1)$ $= \frac{1}{2}$

(b) $A(3)$ $= \frac{1}{2} + 2\cdot 1 + \frac{1}{2} = 3$

(c) $A(6)$ $= \underbrace{\frac{1}{2} + 2\cdot 1 + \frac{1}{2}} + \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}$

$= 3 + 2 - 1$

$= 4.5$

(d) $A'(1)$ $= f(1) = 1$

$$A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$$

(e) $A'(3)$ $= f(3) = 2$

(f) $A'(6)$ $= f(6) = -1$

13. [12 points] Use the following Reimann Sums to approximate the integral $\int_a^b \ln(x)\,dx$

(a) Express the integral $\displaystyle\int_1^2 \ln(x)\,dx$ as the limit of its Right Reimann Sums.

$$\int_1^2 \ln(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \ln(x_i)\,\Delta x\right]$$

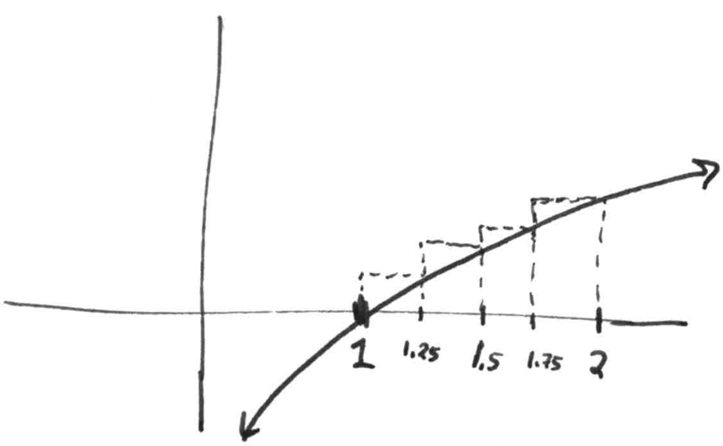
limit 1pt

sum 1pt

area of i^{th} rect: 2pt

(b) Sketch a picture of the Right Sum approximation for $\displaystyle\int_1^2 \ln(x)\,dx$ when $n = 4$.

$$\Delta x = \frac{2-1}{4} = \frac{1}{4} = 0.25$$



correct fn: 1pt

4 slices: 1pt

right rectangles: 2pt

(c) Write out the Right Sum approximation for $\displaystyle\int_1^2 \ln(x)\,dx$ when $n = 4$. You must write out all numbers (endpoints and widths), but you do not need to simplify.

$$R_4 = \ln(1.25)\,0.25 + \ln(1.5)\cdot 0.25 + \ln(1.75)\cdot 0.25 + \ln(2)\cdot 0.25$$

correct formula: 2pt

Δx : 1pt

endpoints : 1pt