Instructions:

- This exam contains 14 pages. When we begin, check you have *one* of each page.
- You will have 2 hours to complete the exam.
- Please **show all work**, and then **write your answer on the line provided.** In order to receive full credit, solutions must be complete, logical and understandable.
- Turn smart phones, cell phones, and other electronic devices off now!

Academic Honesty:

By writing my name below, I agree that all the work which appears on this exam is entirely my own.

I will not look at other peoples' work, and I will not communicate with anyone else about the exam.

I will not use any calculators, notes, etc.

I understand that violating the above carries *serious consequences*, both moral and academic.

Printed Name: Key Signature: ____________

Section: ____________

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	12	12	12	12	10	10	10	12	12	12	12	12	12	150
Score:														

1. [12 points] (a) Let $f(x) = \sin^2(x)$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[(\sin(x))^2\right]$$

$$= 2 \cdot \sin(x) \cdot \cos(x)$$

2pts 2pt

(b) Let $f(x) = \tan(e^x)$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\tan(e^x)\right]$$

$$= \sec^2(e^x) \cdot e^x$$

2pt 2pt

(c) Suppose that $y^2 = x + y$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x+y)$$

$2 \cdot y \cdot$ [redacted] $y' = 1 + y'$ 2pt

$2y \cdot y' - y' = 1$ 1pt

$y'(2y-1) = 1$

$y' = \frac{1}{2y-1}$ 1pt

2. [12 points] Compute the following limits, showing your work.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^2-4}$

Plug in 2 gives $\frac{0}{0}$

$\Rightarrow$ cannot plug in 2.

$$= \lim_{x\to 2} \frac{(x-2)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{(x-2)}{(x+2)} = \frac{0}{4} = 0$$

(b) $\lim\limits_{x\to 0+} e^{\frac{1}{x}} = \infty$

as $x \to 0^+$

$\frac{1}{x} \to \infty$

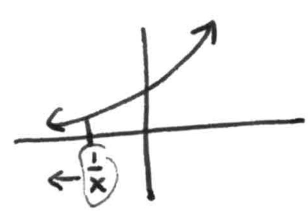
so $e^{\boxed{\frac{1}{x}}} \to \infty$

(c) $\lim\limits_{x\to 0-} e^{\frac{1}{x}} = 0$

as $x \to 0^-$

$\frac{1}{x} \to -\infty$

so $e^{\boxed{\frac{1}{x}}} \to 0$



3. [12 points] Compute the following limits. Remember: you must **show all work**.

(a) $\lim\limits_{x\to\infty} \dfrac{2x^3-16}{3x^3+9}$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(2-\overset{\to 0}{\frac{16}{x^3}}\right)}{\cancel{x^3}\left(3-\underset{\to 0}{\frac{9}{x^3}}\right)}$$

$$= \frac{2}{3}$$

work/factoring – 2pts

answer – 4pts

(b) $\lim\limits_{x\to\infty} \dfrac{\ln(x^2+1)}{x+5}$ $\overset{\to\infty}{}$ $\underset{\to\infty}{}$

2pts

$$\overset{H}{=} \lim_{x\to\infty} \boxed{\frac{\frac{1}{x^2+1}\cdot(2x)}{1}}$$ ← 2pts

$$= \lim_{x\to\infty} \frac{2x}{x^2+1} \quad \begin{matrix}\to\infty\\ \to\infty\end{matrix}$$ ← 2pts

$$\overset{H}{=} \lim_{x\to\infty} \frac{2}{2x} = 0$$

4. [12 points] Let $f(x) = \sqrt{x}$.

(a) Find an equation for the line tangent to the curve $y = \sqrt{x}$ at $a = 4$.

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \quad \leftarrow 2pt$$

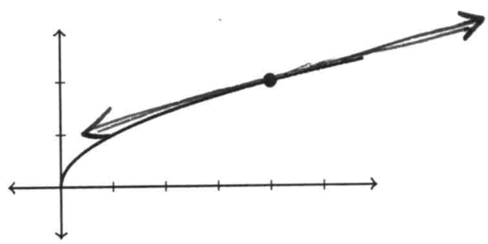
$$f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \leftarrow 1pt$$

$$f(4) = \sqrt{4} = 2 \quad \leftarrow 1pt$$

$$y = m(x - x_1) + y_1 \quad \leftarrow 1pt$$

$$\boxed{y = \frac{1}{4}(x-4) + 2} = \frac{x}{4} + 1 \quad \leftarrow 1pt$$

(b) Sketch the line tangent to the curve at the point $(4, 2)$.



2pt

(c) Find the linearization $L(x)$ of $f(x)$ at $a = 4$, and use it to approximate $\sqrt{4.1}$.

$$L(x) = \frac{1}{4}(x-4) + 2 \quad \leftarrow 1pt$$

1pt for general formula

$$\sqrt{4.1} = f(4.1) \approx L(4.1) = \frac{1}{4}(4.1-4) + 2$$

$$= (.25)(.1) + 2$$

$$= 2.025 \quad \leftarrow 2pt$$

$$= \frac{1}{40} + \frac{2 \cdot 40}{40}$$

$$= \frac{81}{40}$$

5. [10 points] Suppose that a population of bacteria is growing in a petri dish. Suppose also that the first time you look at the dish you count 7 bacteria, and that you count 14 bacteria in the dish 3 hours later.

   (a) Find a formula for the population as a function of the number of hours t since your first measurement.

$P(0) = P_0 = 7$

$P(t) = P_0 \cdot e^{kt}$

$P(t) = 7e^{kt}$

need to find k

Know

$P(\boxed{3}) = 7 \cdot e^{k \cdot \boxed{3}} = 14$

2pt

Solve for k:

$7e^{k \cdot 3} = 14$

$e^{k \cdot 3} = 2$

$k \cdot 3 = \ln(2)$

$k = \dfrac{\ln(2)}{3}$ 2pt

$$\boxed{P(t) = 7 \cdot e^{\left(\frac{\ln(2)}{3} t\right)}}$$ 2pt

   (b) How much time is required for the population to triple in size?

want t s.t.

$3 \cdot 7 = P(t) = 7 \cdot e^{\left(\frac{\ln(2)}{3} t\right)}$

Solve for t

$3 = e^{\frac{\ln(2)}{3} t}$ 2pt

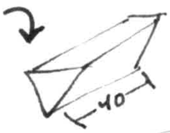
$\ln(3) = \dfrac{\ln(2)}{3} t$

$$\boxed{t = \frac{3 \cdot \ln(3)}{\ln(2)}}$$ 2pt

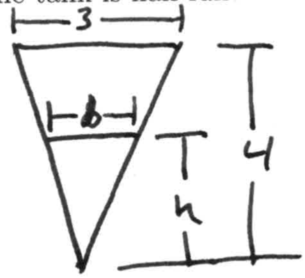
6. [10 points] Suppose there is a 40 m long water trough which is empty at time $t = 0$.

The cross-section of the trough is an inverted triangle ∇ which is 3 m across the top, and is 4 m tall. If the tank is being being filled with water at a constant rate of 5 m^3/s, how fast is the height changing when the tank is half full?

$5\frac{m^3}{s}$



1pt sketch



know

$$\frac{b}{3} = \frac{h}{4}$$

So $b = \frac{3h}{4}$

1pt

$$\boxed{V = l \cdot \tfrac{1}{2}bh} = 40 \cdot \frac{1}{2} \cdot \left(\frac{3h}{4}\right) \cdot h$$

$$V = \frac{30}{2} h^2$$

$$\boxed{V = 15h^2}$$ 2pt

$$\frac{d}{dt} V = \frac{d}{dt}(15h^2)$$

$$\frac{dV}{dt} = 15 \cdot 2h \cdot \frac{dh}{dt}$$

3pt

when $h = \frac{4}{2} = 2$

$$5 = 15 \cdot 2 \cdot 2 \cdot \frac{dh}{dt}$$

So $$\boxed{\frac{dh}{dt} = \frac{5}{15 \cdot 4} = \frac{1}{12}}$$ 2pt

7. [10 points] You are designing a rectangular display box with an open top. The box should have a volume of 8 m^3, and the length of the base should equal two times base's width. The material for the base costs \$3 per square meter, and the material for the sides costs \$1 per square meter. Find the cost of the materials for the cheapest container.

You must **show all work**, but you do not need to verify that the cost is a minimum, and you *do not* need to simplify the final cost.

1 pt

know $V = 8 = l \cdot w \cdot h$

know $l = 2w$

so $8 = (2w)(w)h$

so $\boxed{h = \frac{4}{w^2}}$ 2 pt

Cost = Cost of sides + cost of bottom

2 pt (1 area, 1 cost)

$\boxed{\text{Cost} = 1(2 \cdot l \cdot h + 2 w \cdot h) + 3(l \cdot w)}$

$= \left(2(2w)\left(\frac{4}{w^2}\right) + 2(w)\left(\frac{4}{w^2}\right)\right) + 3(2w) \cdot w$

$= \frac{16}{w} + \frac{8}{w} + 6w^2$

$\boxed{C = \frac{24}{w} + 6w^2}$ 1 pt

$\boxed{C' = \frac{-24}{w^2} + 12w}$ 2 pt

$C' = 0 \iff \frac{24}{w^2} = 12w$

$\iff 2 = w^3$

$\iff \boxed{w = \sqrt[3]{2}}$ 1 pt

$C = \frac{24}{\sqrt[3]{2}} + 6(\sqrt[3]{2})^2$ 1 pt

8. [12 points] Let $f(x) = \dfrac{x^3}{3} - x$

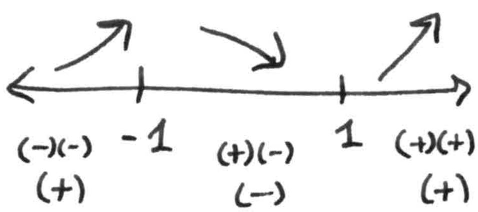
Find the following. If a requested quantity doesn't exist, answer "DNE".

1. The intervals where $f(x)$ is increasing/decreasing. Identify which is which.
2. The intervals where $f(x)$ is concave up/down. Identify which is which.
3. The x value(s) of the local maxima and local minima of f. Identify which is which.
4. The x value(s) of the inflection points of f.

You must show all work.

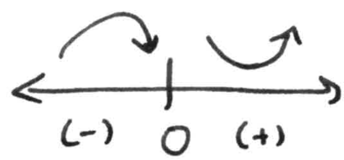
$f'(x) = x^2 - 1$ 1pt

$f'(x) = (x+1)(x-1)$



1pt

$f''(x) = 2x$ 1pt



1pt

f is increasing on $(-\infty, -1) \cup (1, \infty)$

f is decreasing on $(-1, 1)$

1pt for each point/interval

f is concave up on $(0, \infty)$

f is concave down on $(-\infty, 0)$

f has a local max at -1

& has a local min at 1

f has an inflection point at 0

9. [12 points] (a) Compute the general antiderivative for $f(x) = 3x(x^2+1)$

$f(x) = 3x^3 + 3x$ ← 2pt

$$\boxed{F(x) = \frac{3x^4}{4} + \frac{3x^2}{2} + C}$$

2pt 2pt

-1 if No C

(b) Suppose that $f''(t) = -10$, that $f'(0) = 3$ and that $f(0) = 2$. Find a formula for $f(t)$.

$f''(t) = -10$

2pt

$\Rightarrow f'(t) = -10t + C$

$f'(0) = 3 = 0 + C$

$\Rightarrow C = 3$

$f'(t) = -10t + 3$

2pt

$\Rightarrow f(t) = \frac{-10}{2}t^2 + 3t + D$

$f(0) = 2 = 0 + 0 + D$

$\Rightarrow 2 = D$

$$\boxed{f(t) = -5t^2 + 3t + 2}$$

2pt

10. [12 points] Compute the following integrals.

(a) $\displaystyle\int_1^2 \frac{x^2+1}{x}\,dx$

$$= \int_1^2 \left[x + \frac{1}{x}\right] dx$$ 2pt

$$= \left.\frac{x^2}{2} + \ln|x|\right]_1^2$$ 2pt

$$= \left(\frac{2^2}{2} + \ln(2)\right) - \left(\frac{1^2}{2} + \underbrace{\ln(1)}_{=0}\right)$$

$$= \frac{3}{2} + \ln(2)$$ 2pt

(b) $\displaystyle\int_0^3 \frac{1}{1+x}\,dx$

$$\boxed{\begin{array}{l|l} u = 1+x & x=0 \Rightarrow u=1 \\ \frac{du}{dx} = 1 & x=3 \Rightarrow u=4 \\ du = dx & \end{array}}$$ 4 pt

$$= \int_1^4 \frac{1}{u}\,du$$

$$= \left.\ln|u|\,\right]_1^4 = \ln(4) - \ln(1) = \ln(4)$$

1pt 1 pt

11. [12 points] Compute the following integrals.

(a) $\int_0^{\pi} \cos\left(\frac{x}{2}\right)\, dx$

$$\boxed{\begin{array}{l|l} u = \frac{x}{2} & x=0 \Rightarrow u=0 \\ \frac{du}{dx} = \frac{1}{2} & x=\pi \Rightarrow u=\frac{\pi}{2} \\ du = \frac{1}{2}dx & \\ 2du = dx & \end{array}}$$

4pt

$$= \int_0^{\frac{\pi}{2}} 2\cos(u)\, du$$

$$\left(\frac{d}{dx}\sin(x) = \cos(x) \quad \Rightarrow \int \cos(x)\,dx = \sin(x)+C \right)$$

$$= 2\sin(u)\Big]_0^{\pi/2} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0)$$

$$= 2\cdot 1 - 0\cdot 2 = 2$$ [redacted]

2pt

(b) $\int \left[\frac{1}{x} - x\cos(x^2)\right] dx$

$$= \int \frac{1}{x}\,dx - \int x\cdot\cos(x^2)\,dx$$

2pt to split

$$\left(\begin{array}{l} u = x^2 \\ \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \\ \Rightarrow \boxed{\frac{du}{2} = x\,dx} \end{array} \right)$$

2pt for sub

$$= \ln|x| - \int \cos(u)\,\frac{du}{2}$$

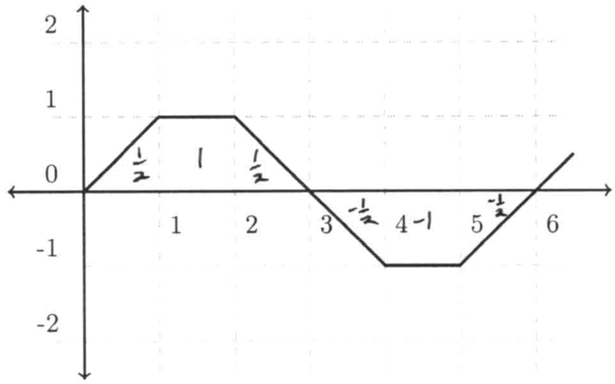
$$= \ln|x| - \frac{\sin(u)}{2} + C$$

$$= \ln|x| - \frac{\sin(x^2)}{2} + C$$

2pt 1pt

-1 if no C

12. [12 points] Suppose that the function $f(x)$ is given by the following graph.



Let $A(x) = \int_0^x f(t)\,dt$. Compute the following

(a) $A(1) = \frac{1}{2}$

(b) $A(3) = 2$

2pts each

(c) $A(5) = \frac{1}{2}$

(d) $A'(1) = f(1) = 1$

$$A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$$

(e) $A'(3) = f(3) = 0$

(f) $A'(5) = f(5) = -1$

13. [12 points] Use the following Reimann Sums to approximate the integral $\int_a^b e^{4x}\,dx$

(a) Express the integral $\displaystyle\int_0^1 e^{4x}\,dx$ as the limit of its Right Reimann Sums.

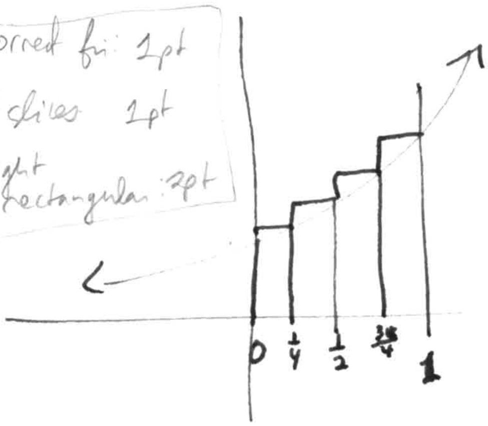
$$\int_0^1 e^{4x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{4\cdot x_i}\,\Delta x$$

$$\left(\begin{array}{l} \text{where } \Delta x = \frac{1-0}{n} = \frac{1}{n} \\ \text{and } x_i = 0 + i\Delta x = \frac{i}{n} \end{array} \right)$$

limit: 1pt
sum: 1pt
rest: 2pt

(b) Sketch a picture of the Right Sum approximation for $\displaystyle\int_0^1 e^{4x}\,dx$ when $n = 4$.

Correct fn: 1pt
4 slices 1pt
Right rectangular: 2pt



$$\Delta x = \frac{1-0}{4} = .25$$

0 .25 .5 .75 1

(c) Write out the Right Sum approximation for $\displaystyle\int_0^1 e^{4x}\,dx$ when $n = 4$. You must write out all numbers (endpoints and widths), but you do not need to simplify.

$$R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= e^{4(.25)}(.25) + e^{4(.5)}(.25) + e^{4(.75)}(.25) + e^{4(1)}(.25)$$

correct formula: 2pt
Δx: 1pt
endpoint: 1pt