Instructions:

- This exam contains 11 pages. When we begin, check you have *one* of each page.
- You will have 70 minutes to complete the exam.
- Please **show all work**, and then **write your answer on the line provided.** In order to receive full credit, solutions must be complete, logical and understandable.
- Turn smart phones, cell phones, and other electronic devices off now!

Academic Honesty:
By writing my name below, I agree that all the work which appears on this exam is entirely my own.
I will not look at other peoples' work, and I will not communicate with anyone else about the exam.
I will not use any calculators, notes, etc.
I understand that violating the above carries *serious consequences*, both moral and academic.

Printed Name: Key Signature: ______________

Section: ______________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	20	10	10	10	10	10	10	10	10	100
Score:										

1. Tricky Derivatives (product and chain rule).

   (a) [5 points] Let $f(x) = \sqrt{4x+1}$. Find the **second** derivative of f.

$$f(x) = (4x+1)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4 \quad = 2 \cdot (4x+1)^{-\frac{1}{2}}$$ 3

$$f''(x) = 2 \cdot \left(\frac{-1}{2}\right) \cdot (4x+1)^{-\frac{3}{2}} \cdot 4$$ 2

$$= \frac{-4}{(4x+1)^{\frac{3}{2}}} = \frac{-4}{\left(\sqrt{4x+1}\right)^3}$$

-1 wrong constant
-1 No Neg

$f''(x) =$ ______________________

   (b) [5 points] Let $f(x) = \tan\left(\cos(x) + \sin(x)\right)$. Find $f'(x)$.

$$f'(x) = \sec^2\left(\cos(x) + \sin(x)\right) \cdot \frac{d}{dx}\left(\cos(x) + \sin(x)\right)$$ 2pt

$$= \sec^2\left(\cos(x) + \sin(x)\right) \cdot \left(-\sin(x) + \cos(x)\right)$$

Both OK

$$= \underbrace{\sec^2}_{1pt}\underbrace{\left(\cos(x) + \sin(x)\right)}_{1pt} \cdot \underbrace{\left(\cos(x) - \sin(x)\right)}_{1pt}$$

(-1) if incorrect simplification

(-2) if distribute tan to sum

$f'(x) =$ ______________________

2. (a) [5 points] Let $f(x) = x^{4x+1}$. Find $f'(x)$.

$$y = x^{4x+1}$$

$$\ln(y) = \ln\left(x^{4x+1}\right) = (4x+1)\cdot\ln(x)$$ 2pt

$$\frac{1}{y}\cdot y' = (4x+1)\cdot\frac{1}{x} + \ln(x)\cdot 4$$ 2pt

$$y' = y\left(\frac{4x+1}{x} + 4\cdot\ln(x)\right)$$

$$\boxed{f'(x) = y' = x^{4x+1}\left(\frac{4x+1}{x} + 4\cdot\ln(x)\right)}$$ 1pt

$f'(x) =$ ______________________

(b) [5 points] Calculate y' if

$$xe^y = y - 1$$

$$\frac{d}{dx}(x\cdot e^y) = \frac{d}{dx}(y-1)$$

$$x\cdot\frac{d}{dx}(e^y) + e^y\cdot\frac{d}{dx}(x) = \frac{d}{dx}(y) - 0$$ 2pt

$$x\cdot e^y\cdot\frac{dy}{dx} + e^y\cdot 1 = \frac{dy}{dx}$$

$$x\cdot e^y\cdot y' - y' = -e^y$$ 1pt

$$y'(x\cdot e^y - 1) = -e^y$$ 1pt

$$\boxed{y' = \frac{-e^y}{x\cdot e^y - 1}}$$ 1pt $\frac{e^y}{1 - x\cdot e^y}$

$y'(x) =$ ______________________

3. [10 points] Remember: you must show all work to earn full credit.

Suppose that a population of bacteria in a petri dish grows at a rate proportional to its size. Suppose that the dish starts out with 100 bacteria in it, and that you count 300 bacteria in it after 2 hours have passed.

10 pts for this

(a) Find an equation for the population as a function of t in hours.

$$\frac{\frac{dP}{dt}}{P} = k \Rightarrow \boxed{\frac{dP}{dt} = k \cdot P(t) \Rightarrow \boxed{P(t) = P_0 \cdot e^{kt}}}$$

2pts for just this

$P(0) = 100$

$P(2) = 300$

$$\boxed{P(t) = 100 \cdot e^{kt}} \leftarrow 5 \text{ pt}$$

$$P(2) = 300 = 100 \cdot e^{k \cdot 2}$$

5 pts

$$3 = e^{k \cdot 2}$$

$$\ln(3) = k \cdot 2 \Rightarrow \boxed{k = \frac{\ln(3)}{2}}$$

$$\boxed{P(t) = 100 \cdot e^{\frac{\ln(3)}{2} t}}$$

(b) What is the doubling time for the bacteria population?

5 pts extra credit for this.

$$200 = 100 \cdot e^{\frac{\ln(3)}{2} t}$$

$$\Leftrightarrow$$

$$2 = e^{\frac{\ln(3)}{2} t}$$

$$\ln(2) = \frac{\ln(3)}{2} t$$

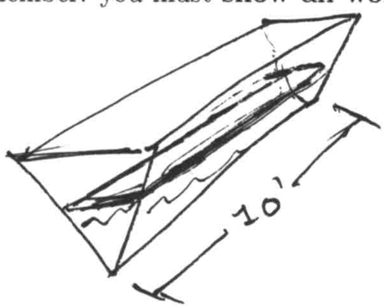
$$\boxed{t = \frac{2 \cdot \ln(2)}{\ln(3)}}$$

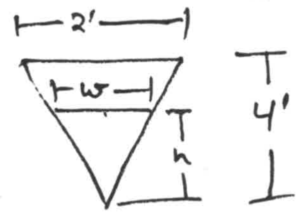
this is the doubling time.

4. [10 points] You are filling an old-fashioned trough with water for your livestock. The trough is 4' tall, 2' wide, and 10' long, and its cross-section is a point-down triangle.

If the trench begins empty, and if you are filling it at the rate of 10 [redacted], what is the rate of change of the height of the water when the trench has 1' of water in it?

Remember: you must **show all work**, including a sketch with appropriate annotations.





1pt for sketch

1 pt

Note: $\frac{w}{2} = \frac{h}{4}$

by similar triangles

So $4w = 2h$

so $w = \frac{h}{2}$

1 pt $\boxed{\text{Volume} = l \cdot \frac{1}{2} w \cdot h}$

$\text{Volume} = 10 \cdot \frac{1}{2} \cdot w \cdot h$

$\text{Volume} = 5 \cdot \frac{h}{2} \cdot h$

2 pt $\boxed{\text{Volume} = \frac{5}{2} h^2}$

3pt

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{5}{2}h^2\right) = \frac{5}{2} \cdot 2h^1 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = 5 \cdot h \cdot \frac{dh}{dt}$$

when $\frac{dV}{dt} = 10$ and $h = 1$

$$10 = 5 \cdot 1 \cdot \frac{dh}{dt}$$

2pt $\Rightarrow \boxed{\frac{dh}{dt} = \frac{10}{5} = 2}$

5. [10 points] Let $f(x) = x \cdot e^x$.

(a) Find a linear approximation $f(x)$ at $a = 1$.

$\boxed{L(x) = f'(a)(x-a) + f(a)}$ 2

$f(a) = f(1) = 1 \cdot e^1 = e$ 1

$f'(x) = \frac{d}{dx}(x \cdot e^x) = x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x)$

$f'(x) = x \cdot e^x + e^x$ 2

$f'(a) = f'(1) = 1 \cdot e^1 + e^1 = e + e = 2e$ 1

$L(x) = 2e(x-1) + e$ 2

$= 2ex - 2e + e = \boxed{(2e) \cdot x - e}$

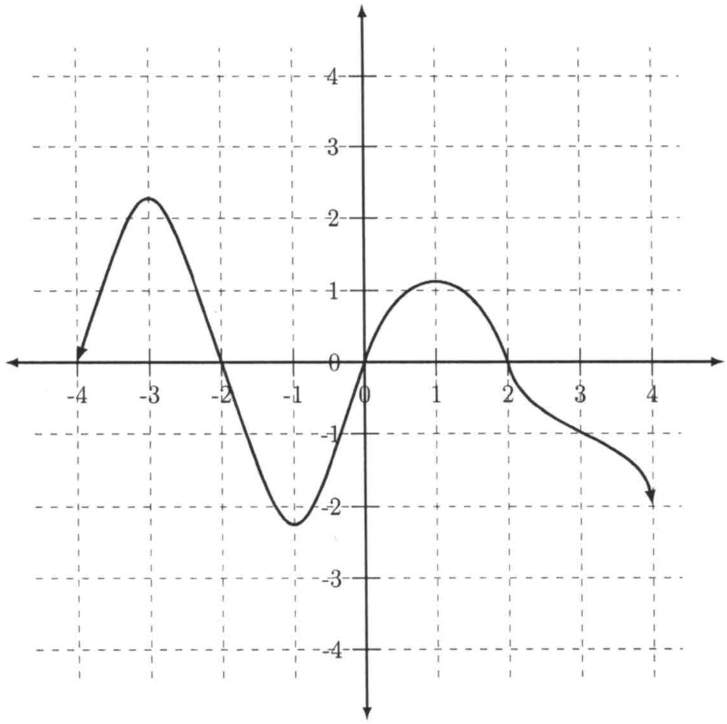
(b) Use this to approximate $f(1.1)$

$f(1.1) \approx L(1.1) = 2e(1.1) - e$ 2

$= (2.2)e - e$

$= (1.2)e$

6. [10 points] Suppose that $f(x)$ is defined using the following graph.



Revised: possible 18 pts

1 per correct part

Find each of the following if they exist. If a requested quantity doesn't exist, answer "DNE".

1. The intervals where f is increasing/decreasing

increasing: $(-\infty, -3) \cup (-1, 1)$

decreasing: $(-3, -1) \cup (1, \infty)$

2. The intervals where f is concave up/down

concave up: $(-2, 0) \cup (2, 3)$

concave down: $(-\infty, -2) \cup (0, 2) \cup (3, \infty)$

3. The local max/min

local max: $x = -3$, $x = 1$

local min: $x = -1$,

4. The absolute max/min

absolute max at -3

absolute min DNE

5. The inflection points

$x = -2, 0, 2, 3$

7. [10 points] Let

$$f(x) = x^3 - 6x^2 + 9x + 1$$

Find the following. If a requested quantity doesn't exist, answer "DNE".

1. the intervals where $f(x)$ is increasing/decreasing
2. the intervals where $f(x)$ is concave up/down
3. the local maxima/minima of f, and
4. the inflection points of f.

You must show all work.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$$

critical points at $x = 3$ and $x = 1$

$f'(x)$ is... (+) on $(-\infty, 1)$: $(-)(-)$; at 1; (−) on $(1, 3)$: $(-)(+)$; at 3; (+) on $(3, \infty)$: $(+)(+)$

↗ ↘ ↗

f is increasing on $(-\infty, 1) \cup (3, \infty)$
f is decreasing on $(1, 3)$
f has local max at $x = 1$
local min at $x = 3$

$$f''(x) = 6x - 12 = 6(x-2)$$

Possible inflection pts at $x = 2$

$f''(x)$ is... $6(-)$ = (−) on the left of 2; $6 \cdot (+)$ = (+) on the right of 2

f is concave down on $(-\infty, 2)$
f is concave up on $(2, \infty)$
f has an inflection point at $x = 2$

8. [10 points] L'Hopital's Rule

(a) $\lim_{x\to 0^+} x \cdot \ln(x)$

$\underset{0}{\downarrow} \quad \underset{-\infty}{\downarrow}$

1 pt

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \quad \begin{matrix}\to -\infty \\ \to \infty\end{matrix}$$

2pt

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \cdot \frac{x^2}{x^2}$$

$$= \lim_{x\to 0} \frac{x}{-1} = \boxed{0}$$

2pt

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x^2} \quad \begin{matrix}\to \infty \\ \to \infty\end{matrix}$

1pt

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2\cdot x} \cdot \frac{x}{x}$$

2pt

$$= \lim_{x\to\infty} \frac{1}{2x^2} = \boxed{0}$$

2pt

10. [10 points] Anti-derivatives

(a) Let $f(x) = 18x^2 - 14x + 9$. Find the general antiderivative of $f(x)$.

Guess: $F(x) = 18 \cdot \frac{x^3}{3} - 14 \frac{x^2}{2} + 9x + C$

Check: [redacted] $\frac{d}{dx}\left(18\frac{x^3}{3} - 14\frac{x^2}{2} + 9x + C\right) = 18x^2 - 14x + 9$ ✓

$6x^3 - 7x^2 + 9x + C$

$F(x) =$ $\frac{18x^3}{3} - 7x^2 + 9x + C$ 5 pts

(b) Suppose that $f''(x) = x^3 + \sin(x)$, that $f'(0) = 0$ and that $f(0) = 1$. Find $f(x)$ exactly.

$f'(x) = \frac{x^4}{4} + (-\cos(x)) + C$

$f'(0) = 0 = \frac{0^4}{4} - \cos(0) + C$

$\cos(0) = C$

$\boxed{C = 1}$

$\boxed{f'(x) = \frac{x^4}{4} - \cos(x) + 1}$

Check
$\frac{d}{dx}\left(\frac{x^4}{4} - \cos(x) + C\right)$
$= \frac{4x^3}{4} - (-\sin(x)) + 0$ ✓

$f(x) = \frac{x^5}{(5)(4)} - \sin(x) + 1 \cdot x + D$

Check
$\frac{d}{dx}\left(\frac{x^5}{20} - \sin(x) + x + D\right)$
$= \frac{x^4}{4} - \cos(x) + 1$ ✓

$f(0) = 1 = \frac{0^5}{20} - \sin(0) + 0 + D$

$\boxed{D = 1}$

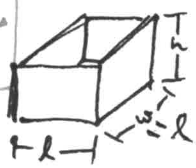
$\boxed{f(x) = \frac{x^5}{20} - \sin(x) + x + 1}$

1833

9. [10 points] You are designing a rectangular display box with an open top. The box should have a volume of 2 m^3, and the length of the base should equal the width of the base. The material for the base costs \$5 per square meter, and the material for the sides costs \$10 per square meter. Find the cost of the materials for the cheapest container.

You must **show all work**, including verifying that this cost is a minimum.

0 pts total if dimensions are guessed



+1 if there is a picture

Note:

Volume $= 2 = l \cdot w \cdot h$

$l = w$

$\Rightarrow 2 = l \cdot l \cdot h$

$\Rightarrow \boxed{h = \frac{2}{l^2}}$ 2pt

Cost = cost of sides + cost of bottom

1 pt for area formula w/ all variables

$= (\text{area of sides})(10) + (\text{area of bottom})(5)$

2 pts for cost formula w/ all variables

$= (l \cdot h + l \cdot h + w \cdot h + w \cdot h)(10) + (l \cdot w)(5)$

because $l = w$ $= (4 \cdot l h) \cdot 10 + 5 \cdot l^2$

2 pts for area in 1 variable

$= 40 \cdot l \cdot h + 5l^2$

$= 40 \cdot l \cdot \frac{2}{l^2} + 5l^2 = \boxed{\frac{80}{l} + 5l^2 = \text{cost}}$ 3pts

Cost when $l = 2$ is

$\frac{80}{2} + 5 \cdot 2^2$

$= 40 + 20$

$= \$60$

$\text{Cost} = \frac{80}{l} + 5l^2$

$\frac{dC}{dl} = \frac{d}{dl}\left(\frac{80}{l} + 5l^2\right) = \frac{-80}{l^2} + 10l$ 2pt

only critical point is when $C'(l) = 0$

$\Leftrightarrow$

$\frac{80}{l^2} = 10l$

$\Leftrightarrow$

$80 = 10l^3$

$8 = l^3 \Leftrightarrow \boxed{l = 2}$ 2pt

Check minimum cost 1pt

1 2 10

$\ominus$ 2 $\oplus$

$\searrow, \nearrow$

$C'(l) = \frac{-80}{l^2} + 10l$

$C'(1) = \frac{-80}{1} + 10 = -70 < 0$

$C'(100) = \frac{-80}{100} + 100 > 0$

there is indeed an absolute min in cost at $l = 2$

18:29