Eg: Solve the System

$$\begin{cases} 2x + 3y = 17 \\ x - y = 6 \end{cases}$$

AKA row reduce

$$\left[\begin{array}{cc|c} 2 & 3 & 17 \\ 1 & -1 & 6 \end{array}\right]$$

IDEA: use r_2 to zero the leftmost nonzero entry of r_1

$r_1^* = r_1 + (-2)r_2$

$$\sim \left[\begin{array}{cc|c} 0 & 5 & 5 \\ 1 & -1 & 6 \end{array}\right] \begin{array}{l} r_1 + (-2)r_2 \\ \text{SAME} \end{array}$$

$$\begin{array}{rl} r_1 & [\,2 \quad 3 \mid 17\,] \\ -2r_2 \; + & [\,-2 \quad 2 \mid -12\,] \\ \hline & [\,0 \quad 5 \mid 5\,] \end{array}$$

flip

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 5 & 5 \end{array}\right]$$

scale r_2

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{l} \text{SAME} \\ \frac{1}{5}r_2 \end{array}$$

IDEA: use r_2 to zero RIGHTMOST nonzero entry of r_1

$r_1^* = r_1 + r_2$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 + r_2 \\ \text{SAME} \end{array}$$

$$\begin{array}{rl} & [\,1 \quad -1 \mid 6\,] \\ + & [\,0 \quad 1 \mid 1\,] \\ \hline & [\,1 \quad 0 \mid 7\,] \end{array}$$

AKA

$$\begin{cases} 1\cdot x + 0\cdot y = 7 \\ 0\cdot x + 1\cdot y = 1 \end{cases}$$

AKA $x = 7$, $y = 1$ is the unique solution

$\boxed{\text{Check by plugging into original eqn}} \Rightarrow \begin{cases} 2\cdot 7 + 3\cdot 1 = 17 \checkmark \\ 7 - 1 = 6 \checkmark \end{cases}$

Eg: Solve the system

$$\begin{cases} x_1 + (-2)x_2 + x_3 = 0 \\ 0 \cdot x_1 + 2x_2 + (-8)x_3 = 8 \\ (-4)x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

AKA Reduce

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

replacing

(use r_1 to kill LEFTMOST nonzero of r_3)

$r_3^* = r_3 + 4r_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] r_3 + 4r_1$$

$$\begin{array}{rcccc} & [-4 & 5 & 9 & | -9] \\ + & [4 & -8 & 4 & | \; 0] \\ \hline \end{array}$$

Scale r_2 (to get leftmost entry $= 1$)

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \tfrac{1}{2} r_2$$

(use r_2 to kill LEFTMOST nonzero of r_3)

$r_3^* = r_3 + r_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \}\text{SAME} \\ \\ r_3 + 3r_2 \end{array}$$

$$\begin{array}{rcccc} & [0 & -3 & 13 & | -9] \\ + & [0 & 3 & -12 & | \; 12] \\ \hline \end{array}$$

$r_2^* = r_2 + 4r_3$

use r_3 to kill RIGHTMOST nonzero coefficient entry of r_2

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad r_2 + 4r_3$$

$$\begin{array}{cccc} [0 & 1 & -4 & | 4] \\ +[0 & 0 & 4 & | 12] \end{array}$$

use r_3 to kill RIGHTMOST nonzero coefficient entry of r_1

$r_1^* = r_1 - r_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad r_1 - r_3$$

$$\begin{array}{cccc} [1 & -2 & 1 & | 0] \\ [0 & 0 & 1 & | 3] \end{array}$$

use r_2 to kill RIGHTMOST nonzero coefficient entry of r_1

$r_1^* = r_1 + 2r_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad r_1 + 2r_2$$

$$\begin{array}{cccc} [1 & -2 & 0 & | -3] \\ +[0 & 2 & 0 & | 32] \end{array}$$

AKA the orig system is equiv. to

$$\begin{cases} x_1 + 0x_2 + 0x_3 = 29 \\ 0x_1 + x_2 + 0x_3 = 16 \\ 0 \cdot x_1 + 0 \cdot x_2 + x_3 = 3 \end{cases}$$

the orig system has the unique solution

$x_1 = 29, \quad x_2 = 16, \quad x_3 = 3$

Check answer by plugging into ORIGINAL system.

Solving an equation the long way:

$$\left(\begin{array}{l} x_1 + 2 \cdot x_2 = 7 \\ 2x_1 + 9\, x_2 = 9 \end{array}\right) \begin{array}{l} A \\ B \end{array}$$

want to get rid of x_1 in eqn B

use A to get rid of leftmost variable in B

$$\Leftrightarrow \left(\begin{array}{l} x_1 + 2 \cdot x_2 = 7 \\ 0 + \overset{5}{} x_2 = -5 \end{array}\right)$$

rescale B

$$\Leftrightarrow \left(\begin{array}{l} x_1 + 2 \cdot x_2 = 7 \\ 0 + 1 \cdot x_2 = -1 \end{array}\right)$$

use B to get rid of RIGHTMOST variable in A

$$\Leftrightarrow \left(\begin{array}{l} x_1 + \overset{0}{} = 9 \\ 0 + x_2 = -1 \end{array}\right)$$

the system has a unique solution!

$$\boxed{\begin{array}{l} x_1 = 9 \\ x_2 = -1 \end{array}}$$

Scratch

$$\begin{array}{l|l} B & 2x_1 + 9\, x_2 = 9 \\ -2A & -(2x_1 + 4x_2 = 14) \\ \hline & 0 + \overset{5}{} x_2 = -5 \end{array}$$

$$\begin{array}{l|l} A & x_1 + 2x_2 = 7 \\ -2B & -(0 + 2x_2 = -2) \\ \hline & x_1 \qquad = 9 \end{array}$$

more concisely

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 5 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & -1 \end{array}\right]$$